THE CHAOS AND STABILITY OF PLANETARY SYSTEMS

GARETT BROWN

1. INTRODUCTION AND BACKGROUND

The number of known exoplanets has increased dramatically over the past 20 years [1]. Along with this increase has come a diversity of planetary arrangements. For stars with multiple exoplanets, the stability of these systems can seem precarious, particularly when they are tightly packed [1]. Yet, for these intractable many-body systems, it seems unlikely that so many would be on the precipice of collapse, especially when the ages of these systems suggest their stability has lasted for millions, or even billions, of years. With gravity as the only fundamental physics governing the dynamics of the system, the stable and persistent nature of these many-body systems calls for further understanding, especially the relationship between resonance and chaos.

1.1. **The Chaotic Nature of Planetary Systems.** The chaotic nature of the three body system governed by gravity was first alluded to by Poincaré and was later characterised in the latter end of the previous century [2]. It has therefore been established that systems with only two degrees of freedom possess an absolute stability of motion. However, for $n > 2$ the motion is always unstable in some sense [3]. As computing power became more widespread, numerical simulations were made of many body systems ($n > 2$) including the solar system and later exoplanet systems. Simulations searching to further constrain orbital elements found many systems to be on the edge of instability [4]. Because of this, resonance was cited as a possible mechanism for maintaining stability on the edge of collapse.

1.2. **Resonance in Celestial Mechanics.** Resonance occurs when the ratio between two characteristics are commensurable, that is, the ratio is that of two integers. There are three kinds of resonances involving orbits in celestial mechanics: *spin-orbital resonance*, the relationship between an planet's spin and its orbit; *secular resonance*, the relationship between the precession in frequencies of planets; and *mean motion resonance*, the relationship between the orbital period of planets [5]. In this paper I will primarily discuss mean motion resonance and I will refer to it simply as resonance. Thus, an example of resonance is the ratio of orbital periods of planetary bodies. For example, in the Jupiter system, Io orbits Jupiter twice, for every one Europa orbit. Thus, Io and Europa share a 2:1 resonance [6].

Resonance plays a key role in the stability and instability of orbiting bodies in our solar system. The two most significant cases are the resonant interactions between Jupiter and the asteroids, and the resonant interactions between Neptune and trans-Neptunian objects[1]. Firstly, there are significant gaps in the distribution of asteroids located between Mars and Jupiter (as can be seen in Fig. 1). The most noteworthy are those possessing small integer multiple resonances with Jupiter. These are known as Kirkwood gaps. These gaps are formed by repeated perturbations from Jupiter on orbit around the Sun. Overtime, the astroids which originally occupied those gaps are shifted to a different orbit. Conversely, trans-Neptunian objects are repeatedly maintained *in* resonant

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[1]Trans-Neptunian objects are Kuiper belt objects with a perihelion closer to the Sun than Neptune's perihelion.

Asteroid Main-Belt Distribution
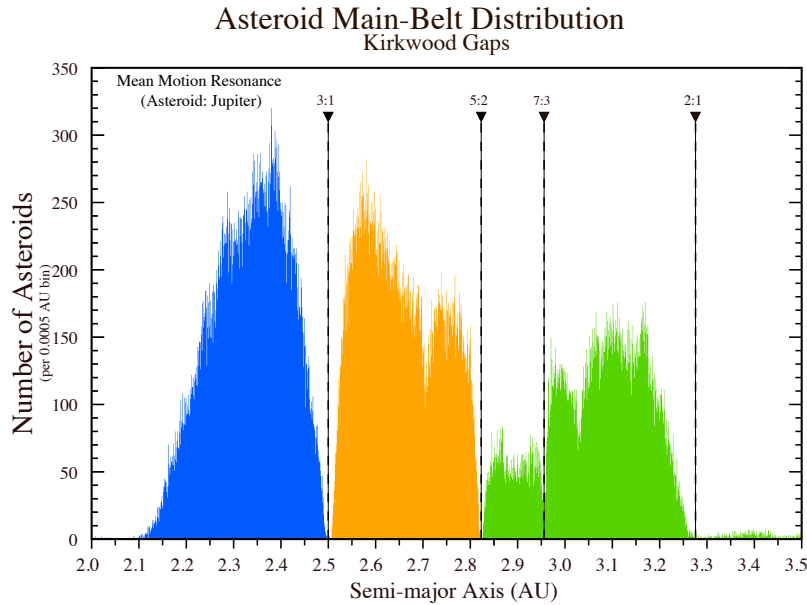### Kirkwood Gaps

FIGURE 1. Gaps in the asteroid belt caused by resonant perturbations from Jupiter. Image from NASA.

bundles with Neptune through consistent interactions. This stablization prevents any possible close encounters that these objects might have as they pass into Neptune's orbit. Thus, both instability (Jupiter moving asteroids out of a particular orbit) and stability (Neptune moving Kuiper belt objects into a particular orbit) are both results of mean motion resonance.

2. MATHEMATICAL AND NUMERICAL METHODS

In order to better understand the interplay between resonance and stability (or instability) we first need to look in the unlikely simplification of a pendulum. The phase space of a pendulum is similar to a Poincaré section of the phase space of resonant planets [3][2]. Investigation of trajectory behaviour near separatrices leads to insight into the development of chaotic regions. And analysis of the chaotic regions with Lyapunov exponents give quantitative results for the expectant stability of planetary systems.

2.1. **The Pendulum.** Further insights and intuition of complicated planetary systems can be developed by comparisons to an ideal system with similar phase space properties — in this case a basic pendulum [8]. In a general way, the dynamics of a simple system posses the *local* structure (or first order approximation) of the dynamics of a pendulum. This is done by averaging the Hamiltonians of the three-body problem (a star with two planets) in the vicinity of the 3:1 and 2:1 mean-motion resonances and then reducing them to the Hamiltonian of a nonlinear pendulum with periodic perturbations [9]. The phase space of a pendulum is described by the equations of motion $\ddot{\theta} = -\mu \sin(\theta)$ and is shown in Fig. 2.

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[2]Unfortunately, this often cited relationship is not straightforward to make and is not trivial to describe. Most of the mathematical machinery which connects the two concepts was developed by Boris Chirikov [3], and was further developed by Ivan Shevchenko [7].
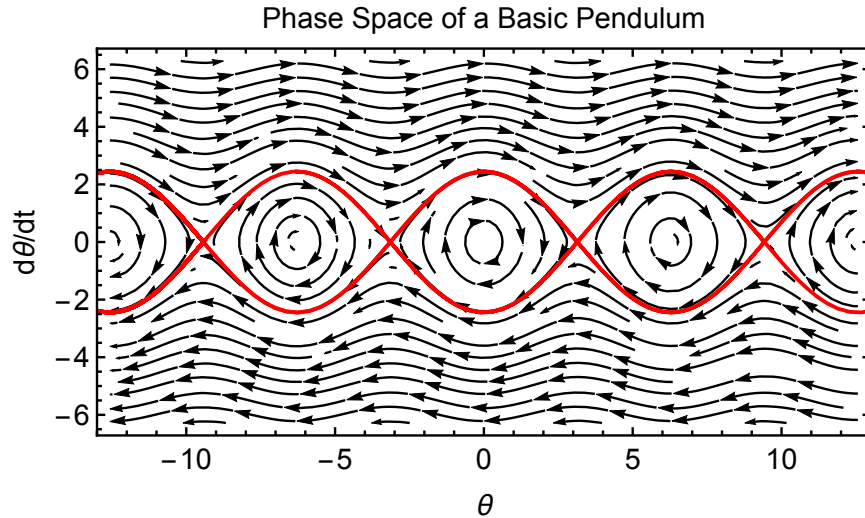
FIGURE 2. The phase space of a pendulum described by $\ddot{\theta} = -\mu \sin(\theta)$, with $\mu = 1.5$. The separatrices between the different phase space basins are show in red.

A linear stability analysis of the pendulum shown in Fig. 2 reveals that there are stable fixed points at the centres of the red peaked ovals $(0, 2\pi, -2\pi, ...)$ surrounded by orbital regions where the pendulum librates around these fixed points. Additionally, there are unstable equilibrium points at each tip of the red peaked ovals $(\pi, -\pi, 3\pi, ...)$. The resonant regions are the libration regions because only for librating orbits is the averaged time derivative of θ equal to zero [8]. Furthermore, there are heteroclinic trajectories which go out from the unstable fix points and into the stable manifold of an adjacent unstable fixed point.

These heteroclinic trajectories follow along separatrices. Consider then that while the motion of a basic pendulum is deterministic and precisely predictable in the longterm, trajectories which lie on or near the phase space separatrices can be relatively impossible to predict due to the sensitive dependence on infinitesimal perturbations to its position or momentum.

2.2. **Separatrices.** Separatrices are phase space trajectories along the stable manifold of saddles which form boundaries between adjacent basins in phase space [10]. In Fig. 2 these separatrices are marked by the solid red lines. The separatrices in the phase space are crucial to developing insight into the predictability of planetary systems. This is becuase the mechanism of the instability is related to the transitions of the system from one resonance to another [3]. This transition inevitably passes through a separatrix region.

In order to better understand how separatrix regions are the mechanism of instability, consider two decoupled resonances (represented in Fig. 3). To obtain a better intuition, we will consider a Poincaré section of the system. A Poincaré section is a two dimensional cross section of a three dimensional phase space [10]. With this in mind, first consider a Poincaré section of two resonances which are sufficiently separated so that each is individually well approximated to first order (Fig. 3 (A)). In this case, we expect well behaved trajectories in most regions of space, perhaps with few anomalies in the regions *very* near the separatrices (as in the pendulum approximation). However, if there are two sufficiently strong resonances, which begin competing to determine the trajectories in the same region of phase space, then significant phase space overlapping can occur (Fig. 3 (B)). In this case, trajectories in these regions of space always lie near separatrix regions [8]. From this brief argument, it may seem that the absence of an overlap of first order resonances provides a

condition for planetary system stability. However, it is only a necessary but not sufficient condition for stability. Indeed one may say also that the overlap criterion gives only an upper limit (in the perturbation strength) for stability [3].

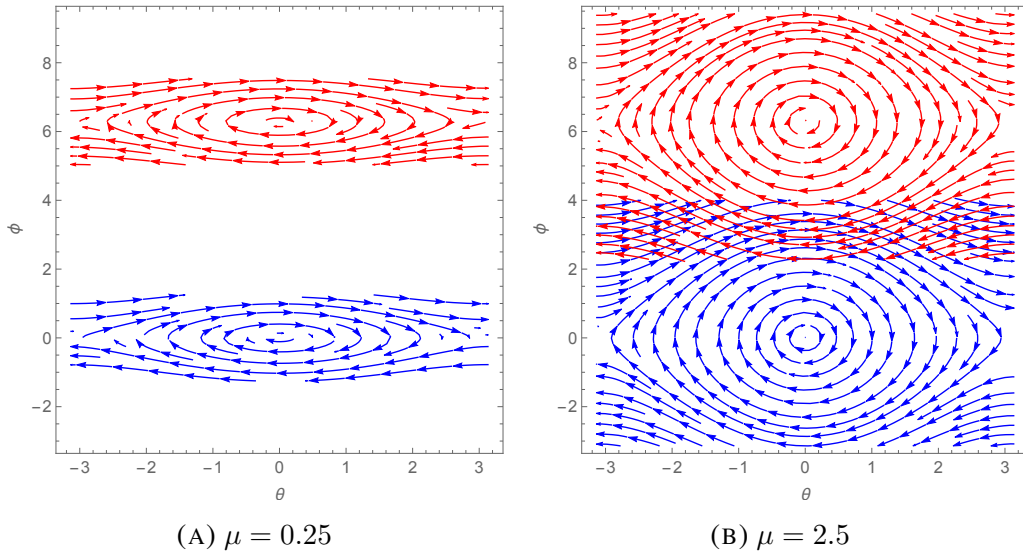(A) $\mu = 0.25$ (B) $\mu = 2.5$

FIGURE 3. Poincaré sections of decoupled resonances modelled as nonlinear pendulums by $\{\dot{\theta} = \phi + \nu, \dot{\phi} = -\mu \sin(\theta)\}$. Here the blue resonance has $\nu = 0$ while the red resonance has $\nu = 2\pi$. When the resonances are well separated, the single resonance approximations are reliable (A). However, when the resonant regions overlap (B), the generation of chaos is expected [8].

Further analysis is required to understand the criteria for stability (or instability and chaos). Even a dimensional reduction of the problem through a physical symmetry argument, such as considering that most of the motion of planetary systems takes place within the orbital plane, is insufficient to glean further details. Full numerical analysis can (and have) been done in order to obtain further insights.

As a case example, consider the work done on the Miranda-Umbriel system[3] by Mel'nikov and Shevchenko [9]. They considered the resonant capture of these two moons by Uranus. They ran multiple simulations with varying eccentricities representing the gradual capture of the two moons. Their results are contained in Fig. 4 where a) has the lowest eccentricity (representing the moons' current positions) and d) has the highest eccentricity (representing the moons' initial capture). As can be seen in their results, as more and more overlap between the two resonances occurs (due to the more eccentric orbits), the regions surrounding the separatrices broaden and become truly chaotic [9].

One of the numerical methods used to calculate the chaotic separatrix regions is a *separatrix map*. This method was originally developed by Boris Chirikov [3] (who called it the whisker map), and was further refined and renamed by Ivan Shevchenko [7]. The separatrix map, which in a lot of ways is similar to the *logistic map* [10], provides a straightforward analytical description of the phase space. Among other things, it also allows one to calculate the locations of resonances and chaos boarders. With this tool, and a stability criteria which depends on separatrix regions,

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[3]Miranda is the fifth moon of Uranus and Umbriel is the second moon of Uranus.
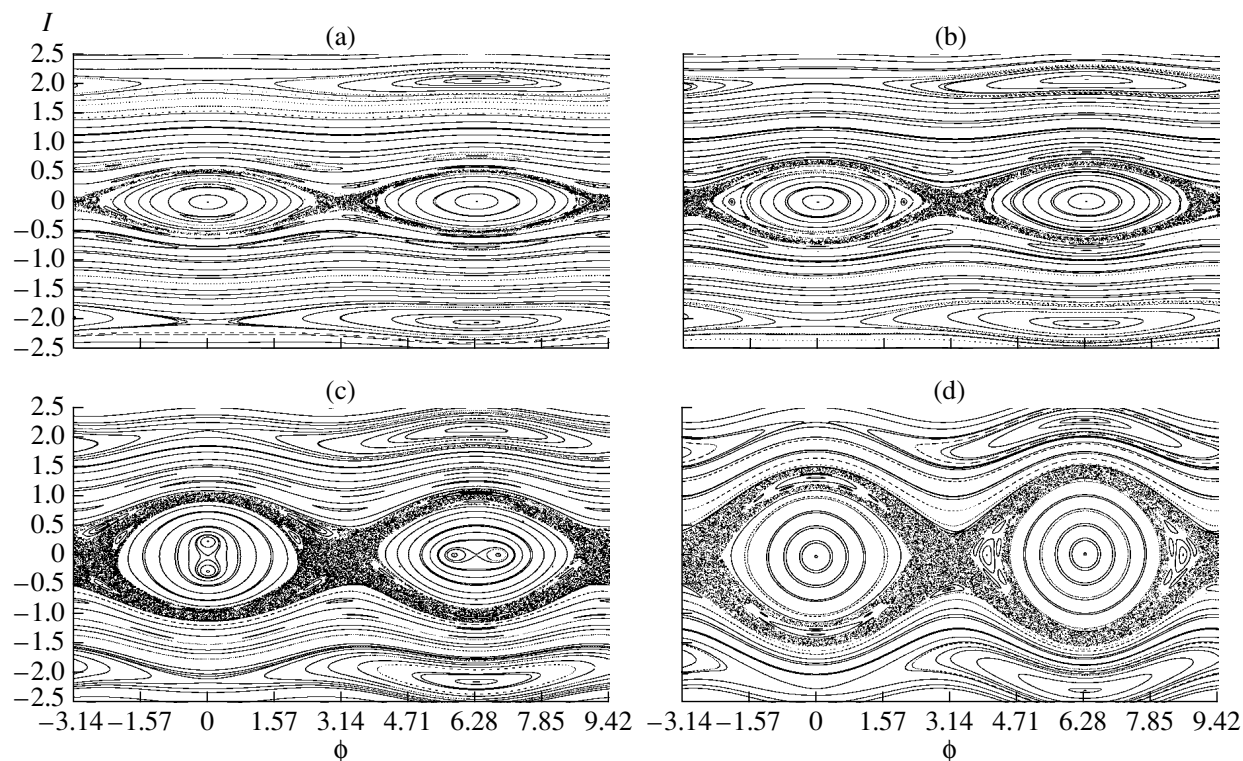
FIGURE 4. Poincaré sections of the phase space of the Miranda-Umbriel system modelling increasing eccentricity (increasing from a to d) [9]. As there is more overlap between the resonances, the chaotic regions broaden.

other tools, such as Lyapunov exponents and Lyapunov times, can be used to help determine the stability (or instability) of planetary systems.

2.3. **Lyapunov Exponents and Lyapunov Times.** Lyapunov exponents are numbers which quantify the sensitive dependance on the initial conditions of a dynamical system. They encapsulate the relative impossibility of long-term prediction in these systems. The idea is to take two very similar initial conditions in phase space and compare their relative separation as a function of time [10]. In mathematical language we define $|\delta(0)|$ as the distance between the initial conditions and $|\delta(t)| \approx |\delta(0)|e^{\lambda t}$ as the distance between the trajectories at time t. Thus, λ would be the Lyapunov exponent characterizing the strength of chaos in a system. There are n Lyapunov exponents in an n dimensional system, however it is customary to choose the largest exponent as characteristic [10] [11].

One particular method for calculating the maximum Lyapunov characteristic exponent (MLCE), λ, is to begin by finding the mean period of rotation of the model pendulum in the chaotic layer [9]. With this period we define η as the ratio of the perturbation frequency to the frequency of small oscillations of the pendulum. Then, the most probable MLCE value is approximated by

(1)
$$\lambda \approx C_h \frac{2\eta}{1 + 2\eta}$$

where the constant $C_h \approx 0.80$. While the essence of the calculation is straightforward, in more complex systems one does need to take into consideration the asymmetric direct and retrograde rotations, as well their dependance on the relative strength of the perturbation amplitude [9].

Once a characteristic Lyapunov exponent λ has been computed, its inverse is on the order of the characteristic Lyapunov time $t_\lambda \approx \mathcal{O}(1/\lambda)$. In particular [10], if d represents the allowed drift[4] in our system and $|\delta(0)|$ is the uncertainty in the measured position of planets, then our prediction power is limited to

$$(2) \qquad\qquad t_\lambda \approx \mathcal{O}\left(\frac{1}{\lambda}\ln\frac{d}{|\delta(0)|}\right) \ .$$

While mathematically equivalent, the advantage of describing chaotic systems with Lyapunov times rather than Lyapunov exponents is the intuition that is obtained because of our natural familiarity with the units of time.

Lyapunov times provide a simple metric for estimating the stability of planetary systems and assist in understanding whether or not we might expect planetary systems to be on the precipice of collapse. The time scales of stability for many well known systems and newly discovered systems have ranged greatly. For example, the time of practical stability of the solar system is on the order of millions of years (5×10^6 yrs) [11], while analysis of some of the proposed exoplanet systems is on the order of hundreds or thousands of years [1]. Other minor systems, such as the Miranda-Umbriel system referenced to earlier, are predicted to be stable on the order of 50 to 100 years.

3. CONCLUSION

With the discovery of so many new exoplanetary systems it becomes imperative to consider whether or not the proposed configurations are stable or even feasible. Our expectations that planetary systems are likely to be stable on the order of stellar lifetimes (billions of years), imply that proposed exoplanet systems with stability timescales of mere hundreds of years are either incorrectly described, or new explanations for their current configurations are needed [1].

Mean-motion resonance is often a characteristic of many body planetary systems which guides our ability to determine the stability or instability of planetary systems. This resonance enables an intuitive nonlinear pendulum approximation which allows us to focus our analysis on the chaotic separatrix regions. These regions can be efficiently and effectively calculated using the separatrix map. Within these regions characteristic Lyapunov exponents and Lyapunov times and be calculated providing a intuitive metric for describing the expected stability (or instability) of these systems.

While this method of calculating intuitive timescales is convenient, the approximations made imply that the results are more of a guideline than an actual verdict. The other resonances alluded to earlier, such as spin-orbit resonance and secular resonance also contribute to a system's dynamics. Fundamental frequency modulation based on changes in secular resonance can also add additional insight and provides an additional metric for chaos. Additionally, in densely packed systems tides and other general relativity effects may also need to be included in the analysis. Both now and in the future, these methods will continue to assist in classifying and confirming the discovery of many more exoplanet systems. [12]

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[4]Drift in the sense that d represents the allowed tolerance of displacement from the supposed orbit.

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