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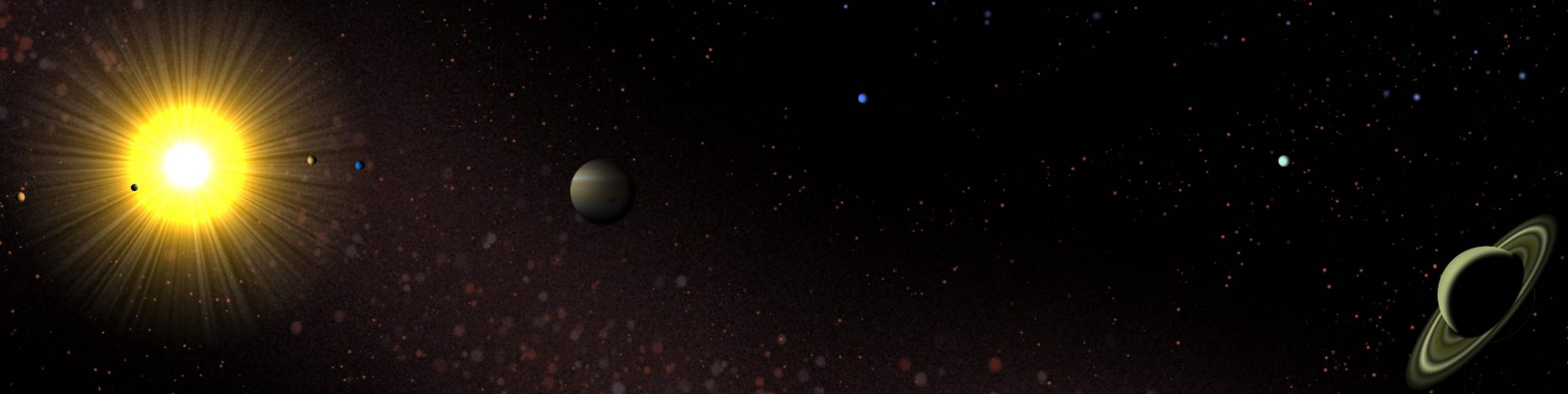
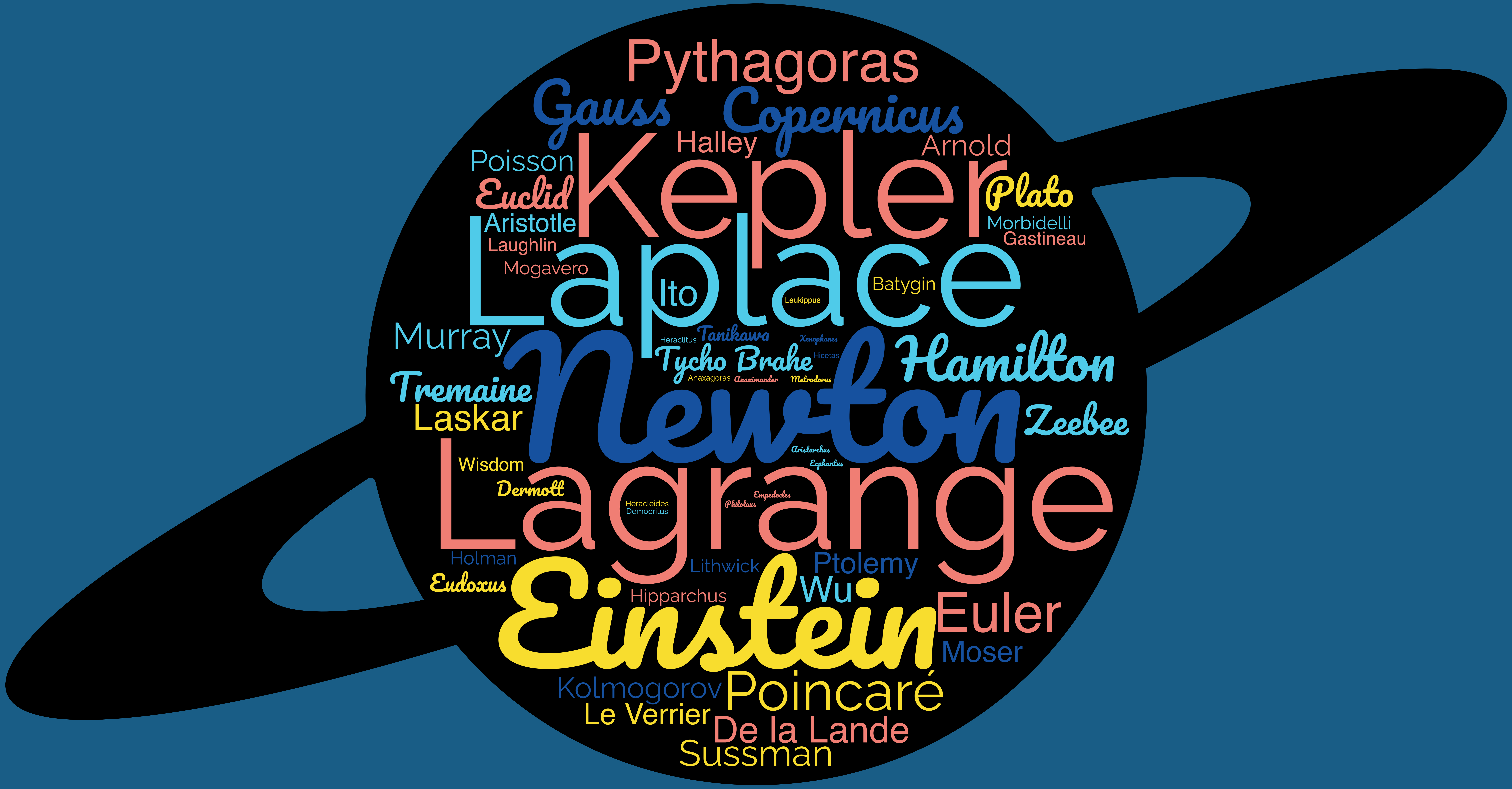


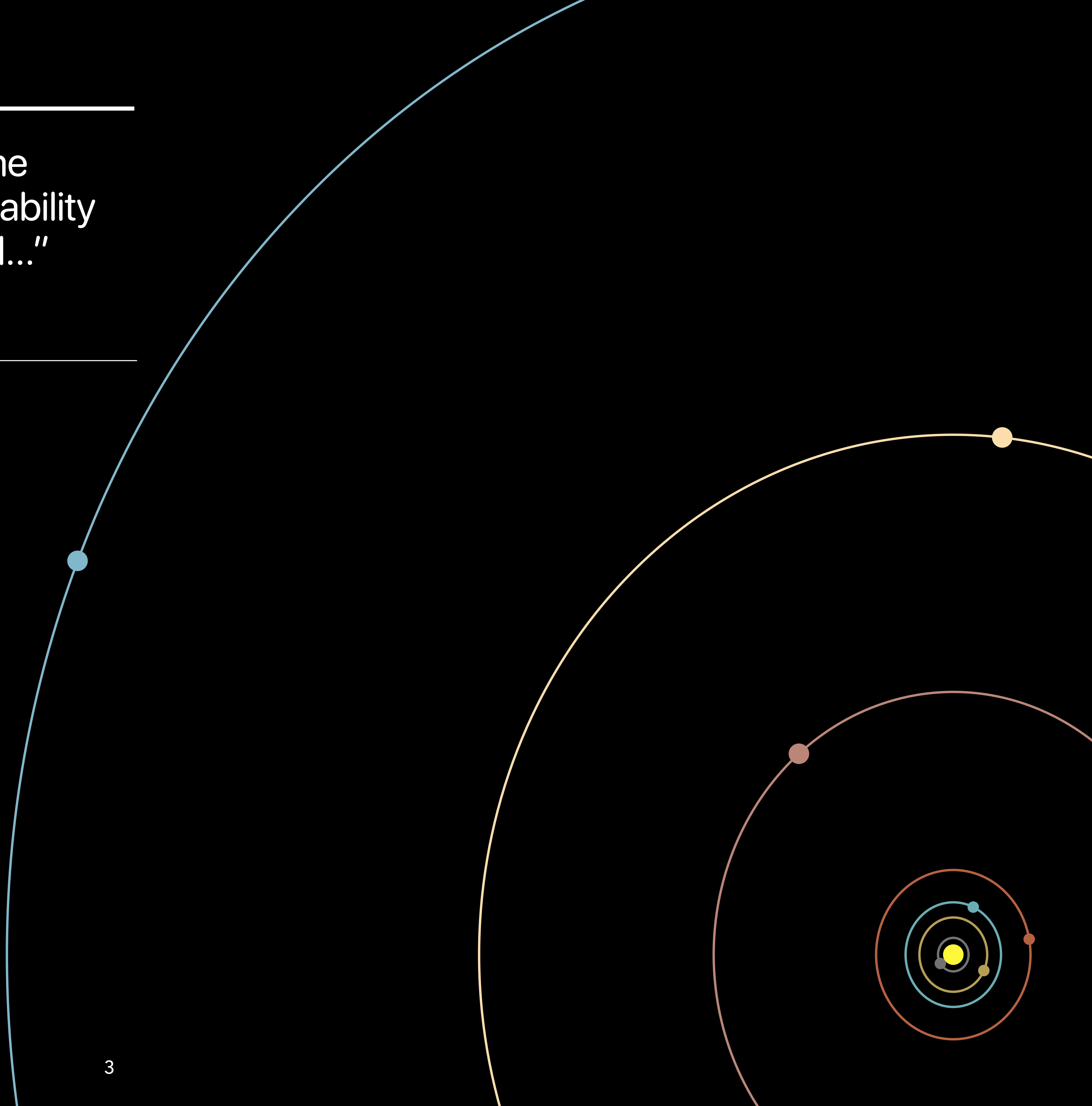
On the Long-term Stability of the Solar System

GARETT BROWN, HANNO REIN


Pythagoras
Gauss
Copernicus
Halley
Arnold
Poisson
Kepler
Euclid
Plato
Aristotle
Morbidelli
Gastineau
Laughlin
Laplace
Mogavero
Ito
Batygin
Leukippus
Murray
Heraclitus
Tanikawa
Xenophanes
Tycho Brahe
Hicetas
Hamilton
Anaxagoras
Anaximander
Metrodorus
Tremaine
Newton
Laskar
Zeebee
Aristarchus
Ecphantus
Wisdom
Dermott
Empedocles
Heracleides
Democritus
Philolaus
Lagrange
Holman
Lithwick
Ptolemy
Eudoxus
Wu
Hipparchus
Einstein
Euler
Moser
Kolmogorov
Poincaré
Le Verrier
De la Lande
Sussman

"Those who are interested... must feel some astonishment at seeing how many times the stability of the Solar System has been demonstrated..."

H. Poincaré (1897)



The Problem

- Universal Law of Gravitation
- Closed solution for 2-bodies
- No analytical solution for 3+ bodies over infinite time.

$$\mathbf{F}_{21} = \frac{Gm_1m_2}{|\mathbf{r}_{21}|^2}\hat{\mathbf{r}}_{21}$$

$$\ddot{\mathbf{r}}_i = \sum_{j=1}^{N} m_j \frac{\mathbf{r}_j - \mathbf{r}_i}{\left|\mathbf{r}_j - \mathbf{r}_i\right|^3}, \quad j \neq i$$

Approaches:

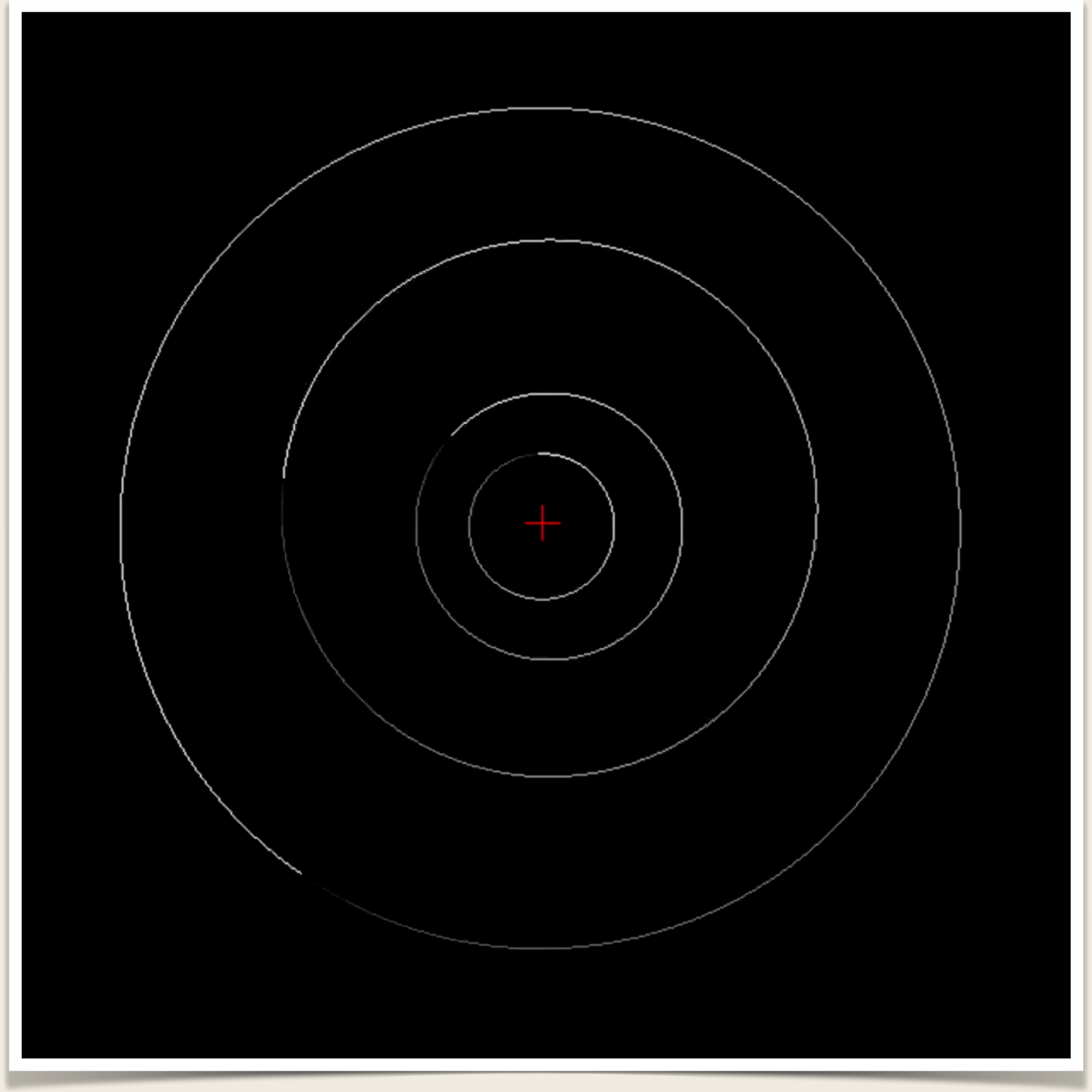
1. Perturbative Expansions
2. Numerical Integrations

$$\mathscr{H} = T + U = \sum_{i=0}^{N-1} \frac{p_i^2}{2m_i} + \sum_{i=0}^{N-1} \sum_{j=i+1}^{N-1} - \frac{Gm_im_j}{|r_i - r_j|}$$

1: Perturbative Expansions

Average over the fast timescales and expand with perturbation theory...

- Effectively assume the planets are interacting rings of mass
- Mutual interaction causes the orbits to precess and deform
- Linearized equations ⇒ eigenvalues (secular frequencies/modes)
- Quasi-periodic functions



2: Numerical Integrations

- Split Hamiltonian into "drift" Kepler step ($\mathscr{A}$) and "kick" Interaction step ($\mathscr{B}$)
- Discretize time step, "drift-kick-drift"
- Compute one step at a time
- Time-step needs to be smaller than smallest timescale in the problem
- Need $> 10^{12}$ steps until Sun's red giant phase

$$\mathscr{H} = T + U = \sum_{i=0}^{N-1} \frac{p_i^2}{2m_i} + \sum_{i=0}^{N-1} \sum_{j=i+1}^{N-1} - \frac{Gm_i m_j}{|r_i - r_j|}$$

$$\mathscr{H} = \mathscr{A} + \mathscr{B}$$

$$\mathscr{A} = \sum_{i=0}^{N-1} \frac{p_i^2}{2m_i} + \sum_{i=1}^{N-1} - \frac{Gm_i m_0}{|r_i - r_0|}$$

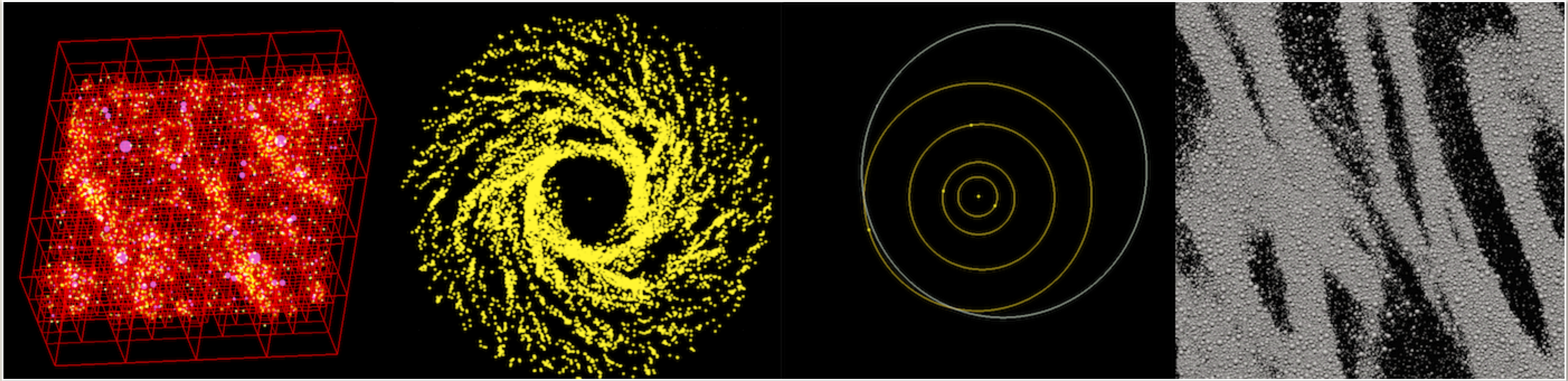
$$\mathscr{B} = \sum_{i=1}^{N-1} \sum_{j=i+1}^{N-1} - \frac{Gm_i m_j}{|r_i - r_j|}$$

REIN, LIU (2012) DOI: 10.1051/0004-6361/201118085; TAMAYO ET AL. (2020) DOI: 10.1093/MNRAS/STZ2870

REBOUND & REBOUNDx

- N-body integrator
- Flexible and customizable
- Easy Python API

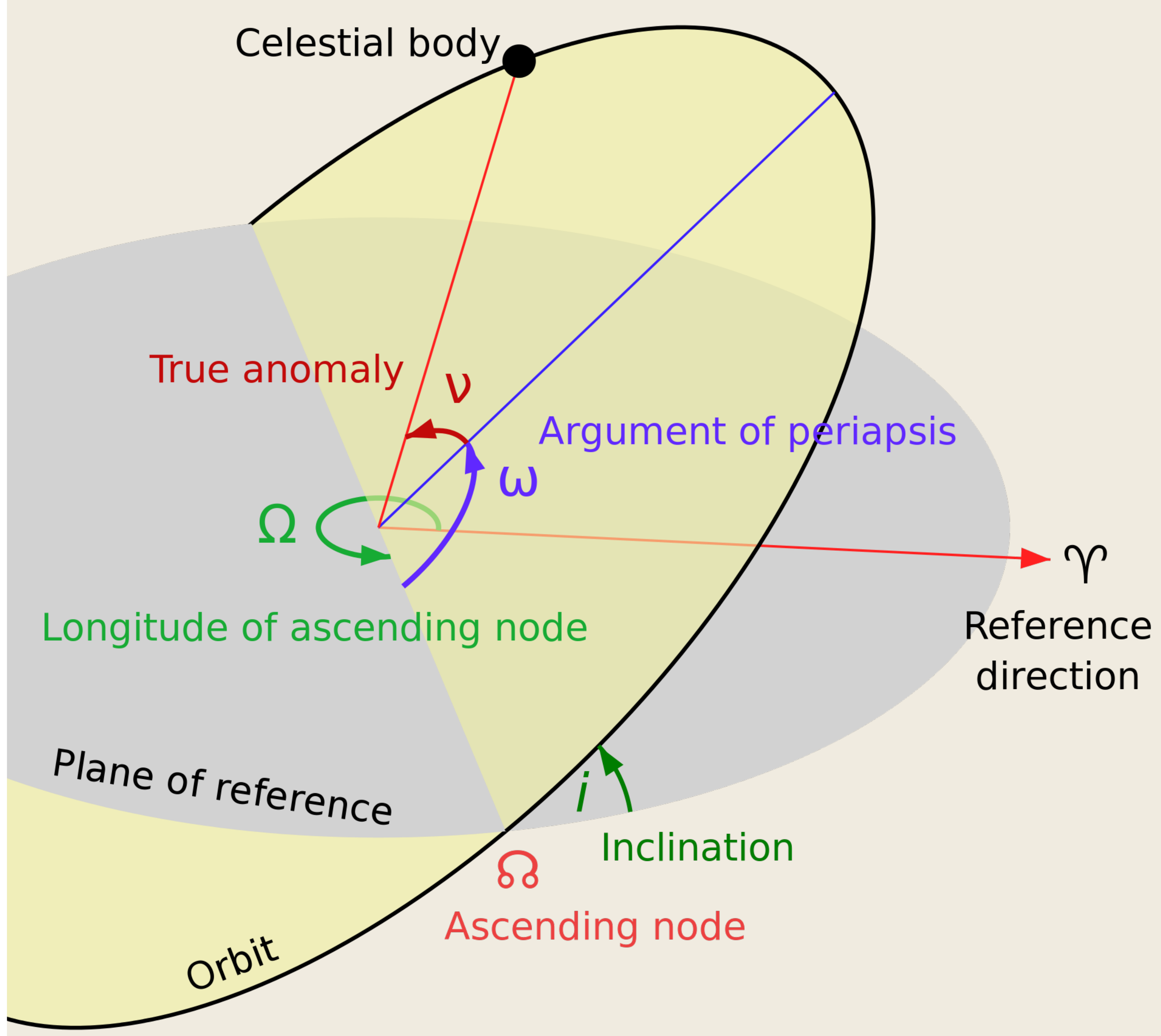
- REBOUNDx provides additional effects, like precession from general relativity



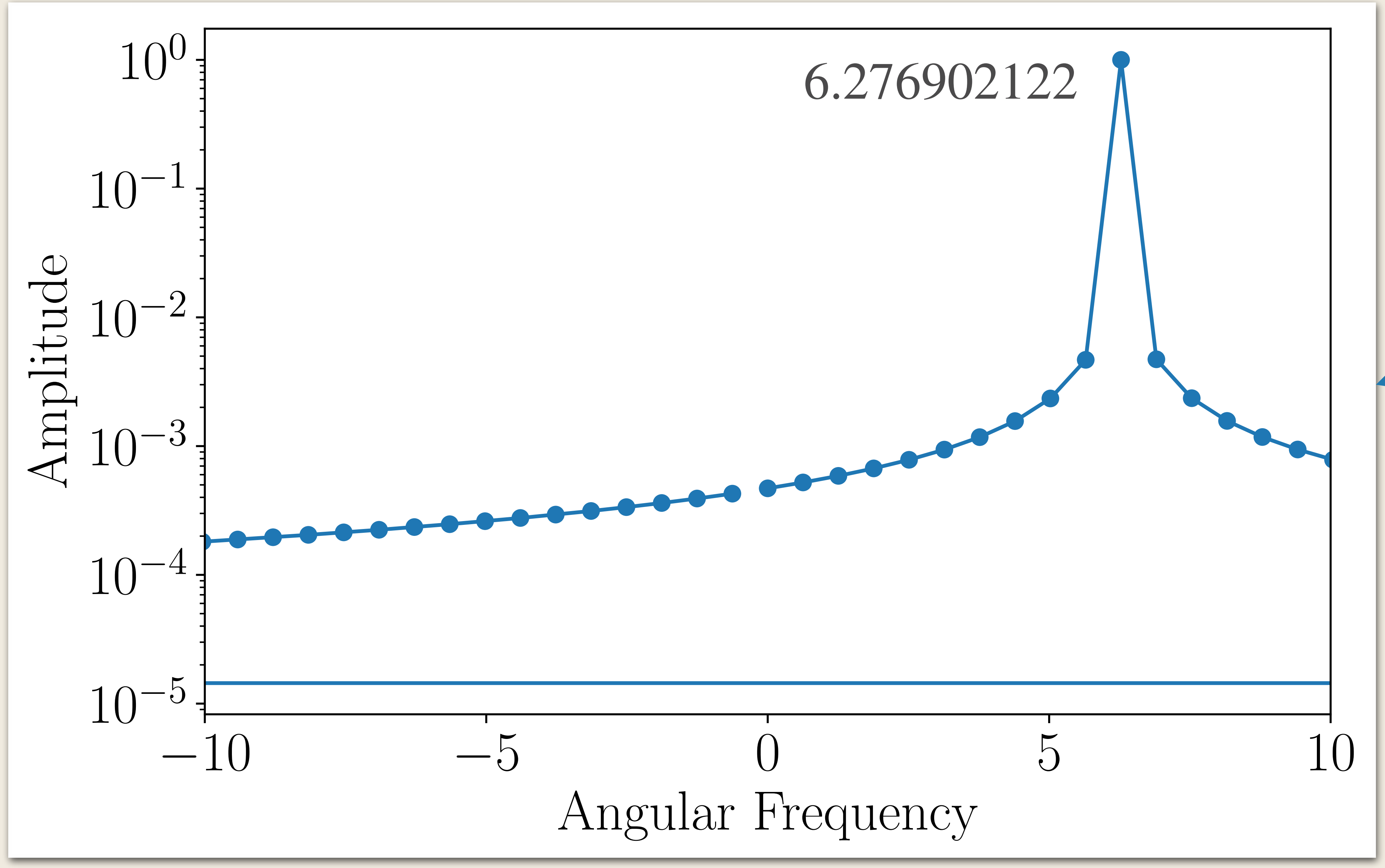
Secular Modes

Integrate over millions of years, then find the modes using Modified Fourier Transform (MFT)

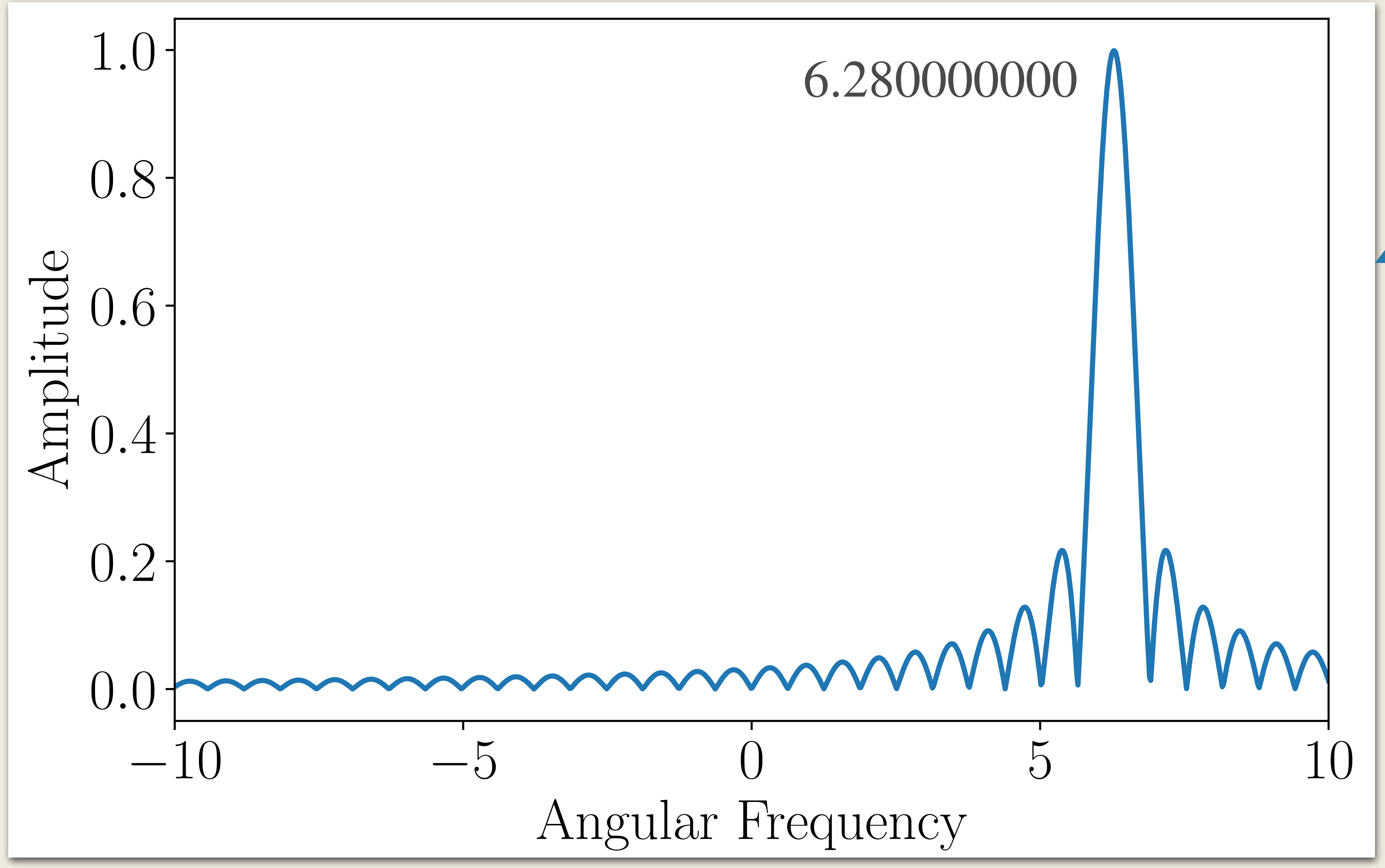
- Numerically integrate
- Sample frequently
- Compute the complex eccentricities and inclinations:
  - $z_j(t) = e_j(t)\ \exp[\mathrm{i}\varpi_j(t)]$
  - $\zeta_j(t) = i_j(t)\ \exp[\mathrm{i}\Omega_j(t)]$
- Compute FMFT of $z_j(t)$ and $\zeta_j(t)$.



Wikipedia/Orbital_elements



FFT



MFT

LASKAR (1990) DOI: 10.1016/0019-1035(90)90084-M
ŠIDLICHOVSKÝ & NESVORNÝ (1996) DOI: 10.1007/BF00048443

Modified Fourier Transform (MFT)

- Contrived Example
- $y(t) = \exp(6.28\mathrm{i}\, t)$
- $t \in [0,10],\ N = 1000$
- $\phi(\nu) = \langle y(t), \exp(\mathrm{i}\nu t) \rangle$
- Maximize $\phi(\nu)$ in the region of ν.

LASKAR (1990) DOI: 10.1016/0019-1035(90)90084-M; ŠIDLICHOVSKÝ & NESVORNÝ (1996) DOI: 10.1007/BF00048443

Secular Modes

Integrate over millions of years, then find the modes using Modified Fourier Transform (MFT)

- Solar System fundamental frequencies
- Obtained with modified Fourier transform (MFT)
- 20 Myrs for the inner planets
  50 Myrs for the outer planets

"/yr	Rein & Brown (2019)	Laplace-Lagrange
g1	**5.589**	**5.462**
g2	**7.422**	**7.347**
g3	**17.359**	**17.332**
g4	**17.914**	**18.006**
g5	**4.258**	**3.733**
g6	**28.246**	**22.512**
g7	**3.088**	**—**
g8	**0.673**	**—**

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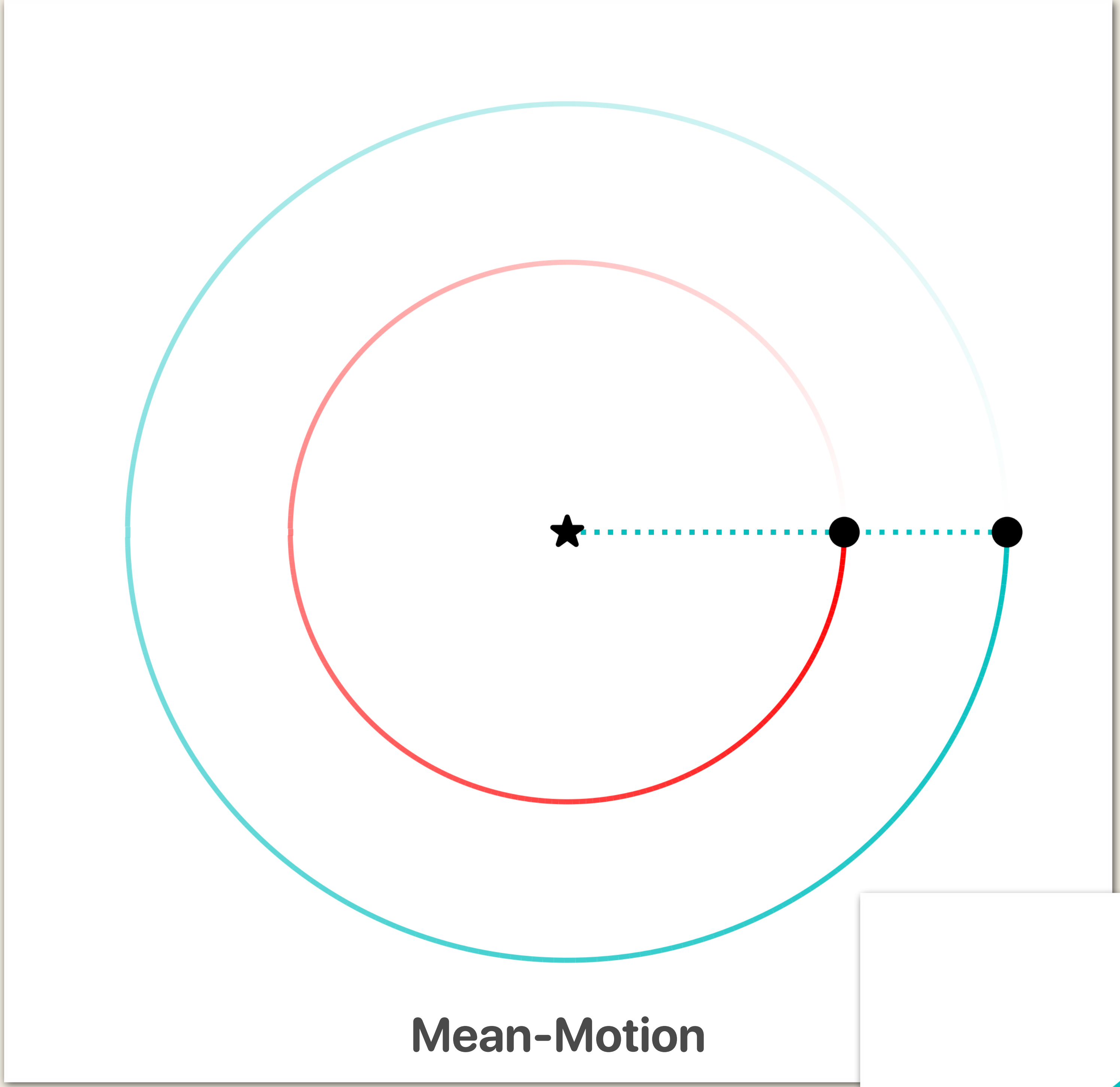
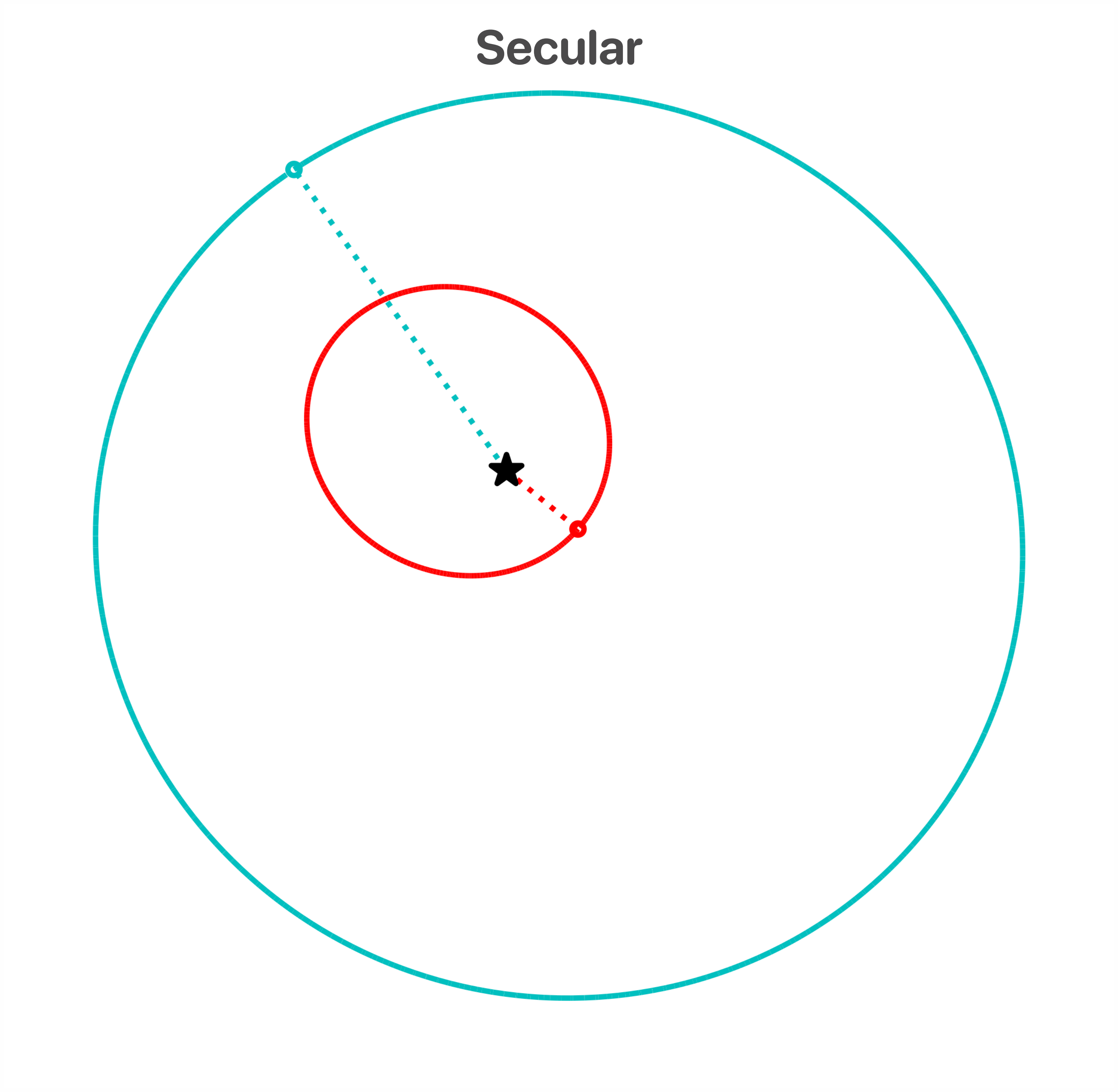




MURRAY & DERMOTT (1999); TREMAINE (2023)

Secular Resonance

- Secular resonances occur when the precessional frequencies of an orbit are similar to another in the system
- Mean-motion resonances occur when the orbital period of one planet is a small integer ratio of another planet's period

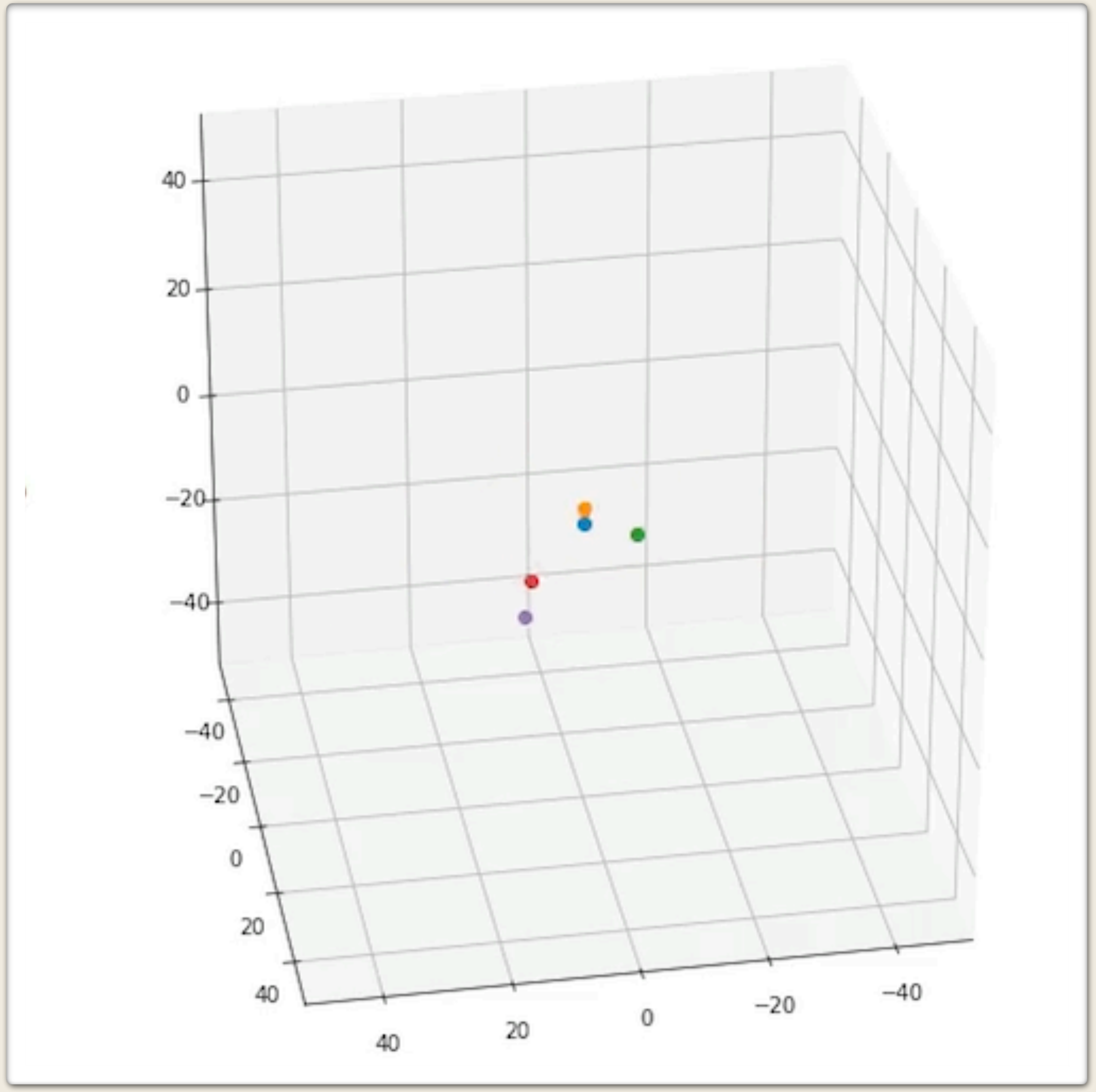
LITHWICK & WU (2011) DOI: 10.1088/0004-637X/739/1/31

Secular Resonance

Secular Resonance

Stellar Flybys

- A passing star, external to the system of interest
- Often very distant and unnoticeable
- Can be very violent and destructive
- Is there a middle ground?

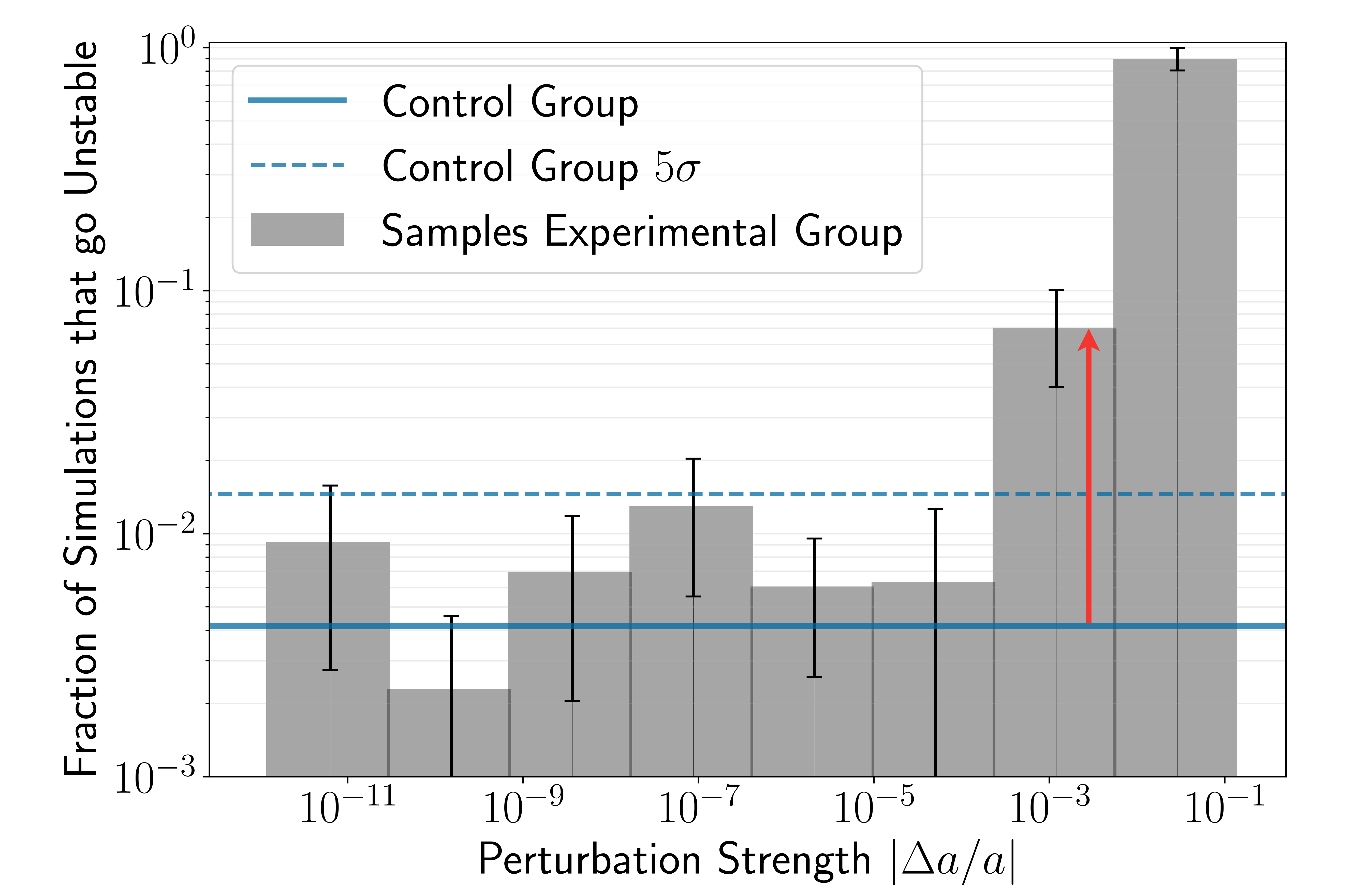


BROWN & REIN (2022) DOI: 10.1093/MNRAS/STAC1763

Experimental Setup: Stellar Flybys

- 2,880, 4.8 Gyrs solar system simulations:
  - 960 control sims, 1920 experimental sims
- NASA JPL Horizons, J2000 epoch
- REBOUND N-body integrator
- REBOUNDx with `gr_potential`
- WHCKL integrator
- `dt = 8.062` days

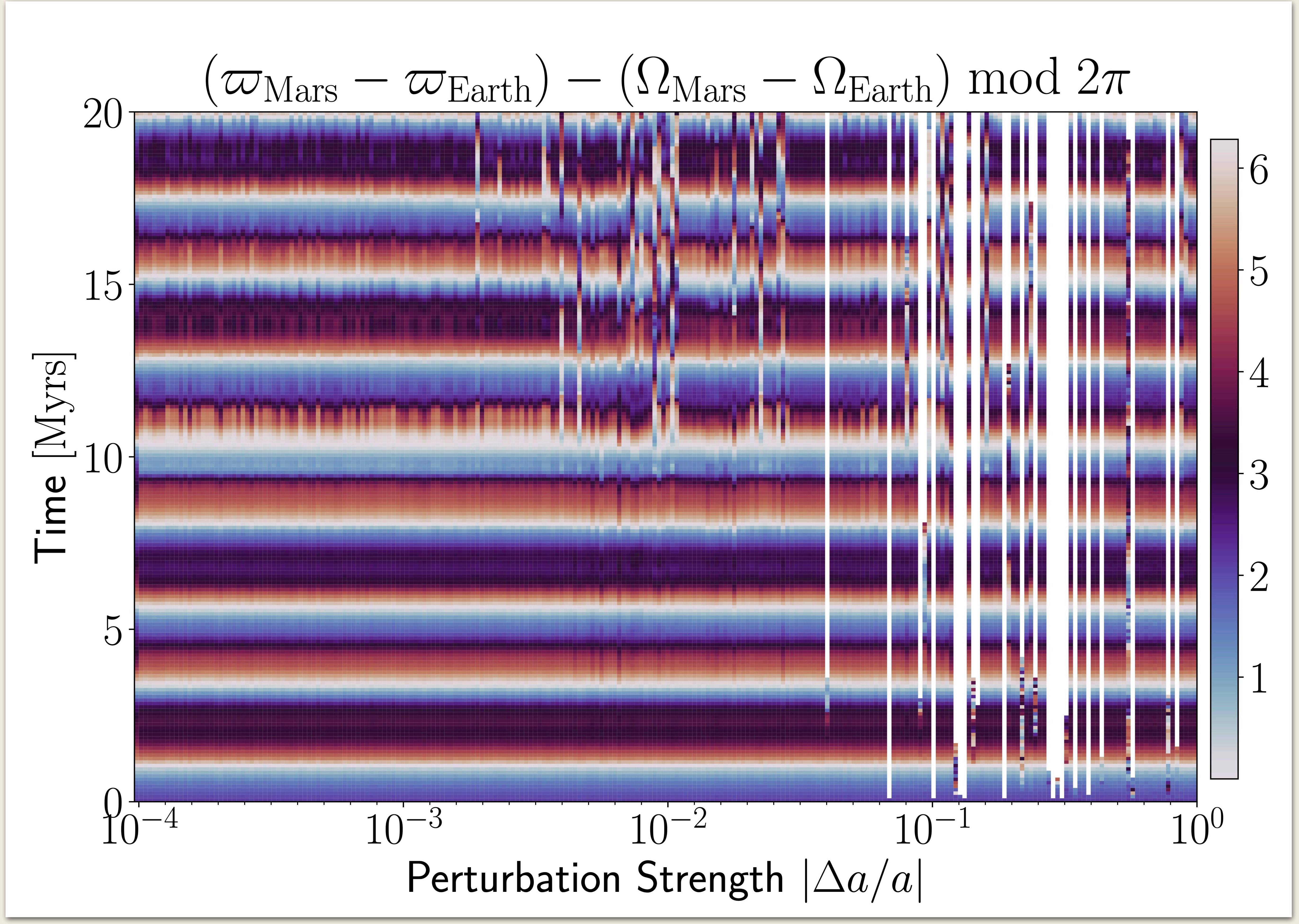
Stellar flybys affect Neptune's orbit more than the other planets



BROWN & REIN (2022) DOI: 10.1093/MNRAS/STAC1763

Stellar Flybys and stability

- 0.1% relative change to Neptune's semi-major axis increases instability within 5 billion years by one order of magnitude.



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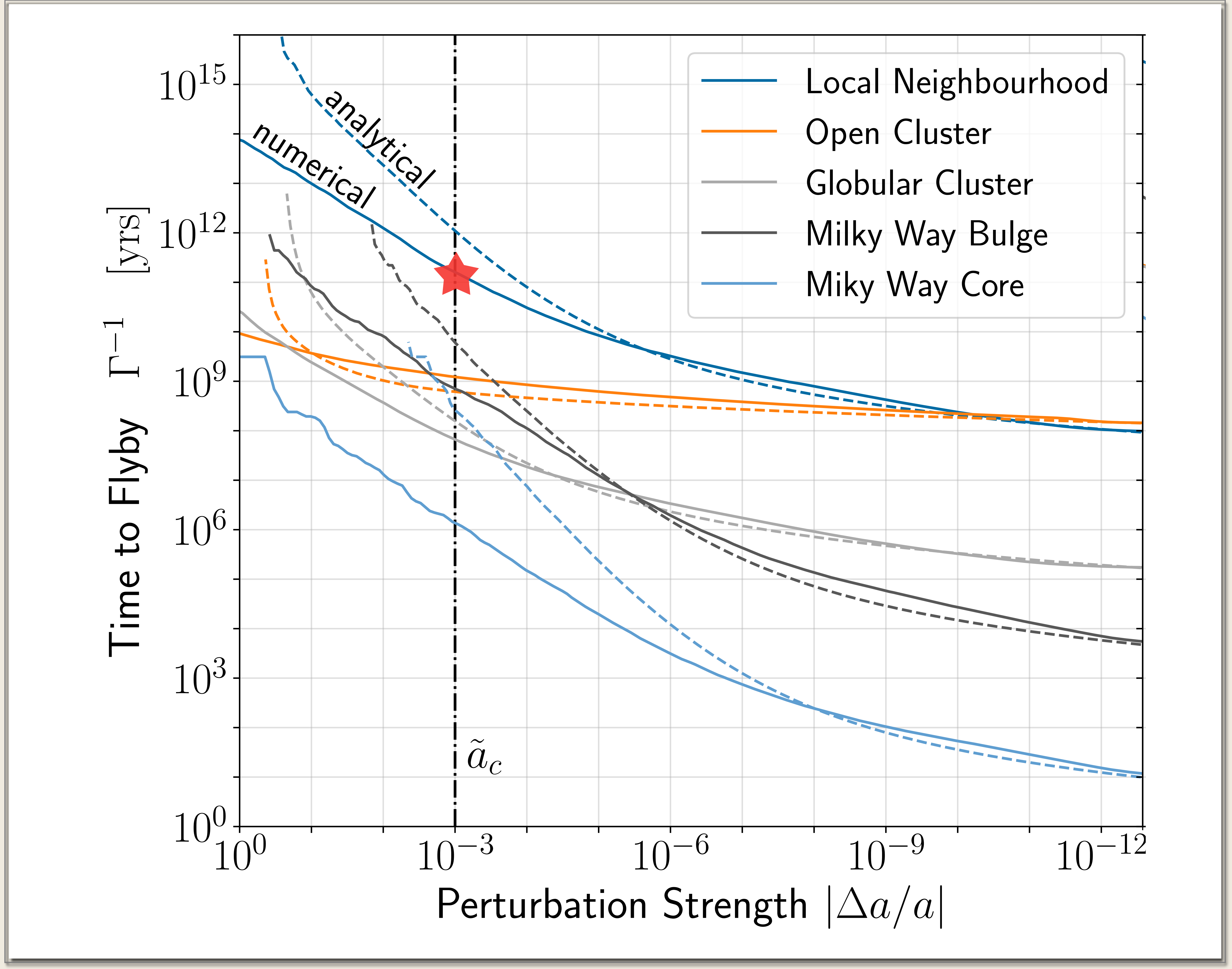
Stellar Flybys and stability

- Changes to Neptune's orbit effect the inner solar system secular state
- Angular momentum is exchanged between the planets on secular timescales



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Stellar Flybys and stability

- A flyby of this magnitude is not likely to pass the solar system for 100 billion years

Summary

- Modern computation provides insights into Newton's centuries old problem.
- The secular diffusion of Mercury through phase space leads to instability when Mercury enters a linear secular resonance with Jupiter.
- Even weak stellar flybys can increase the chance of solar system instability, over the subsequent 5 billion years.