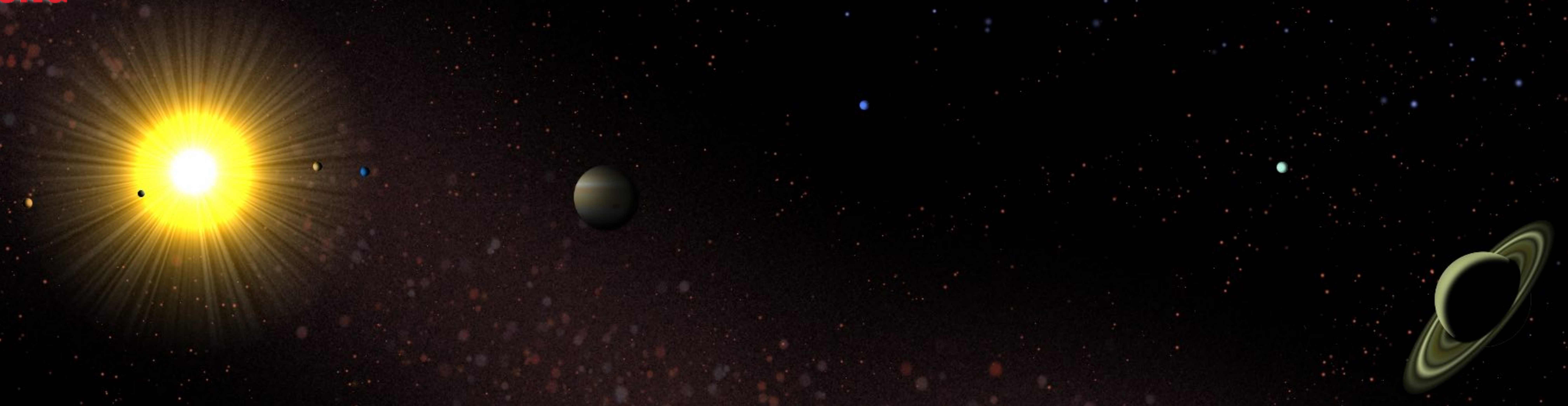
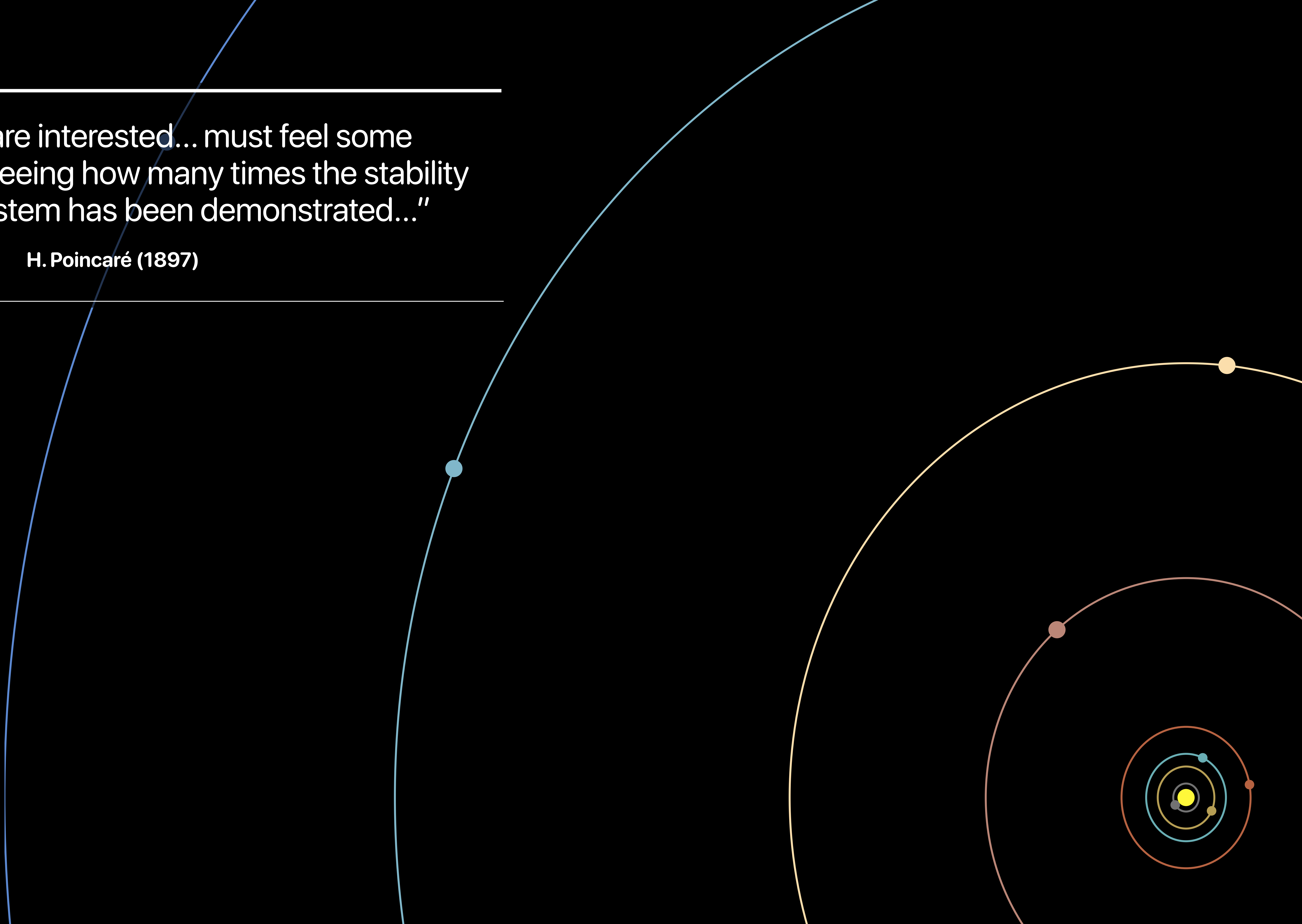


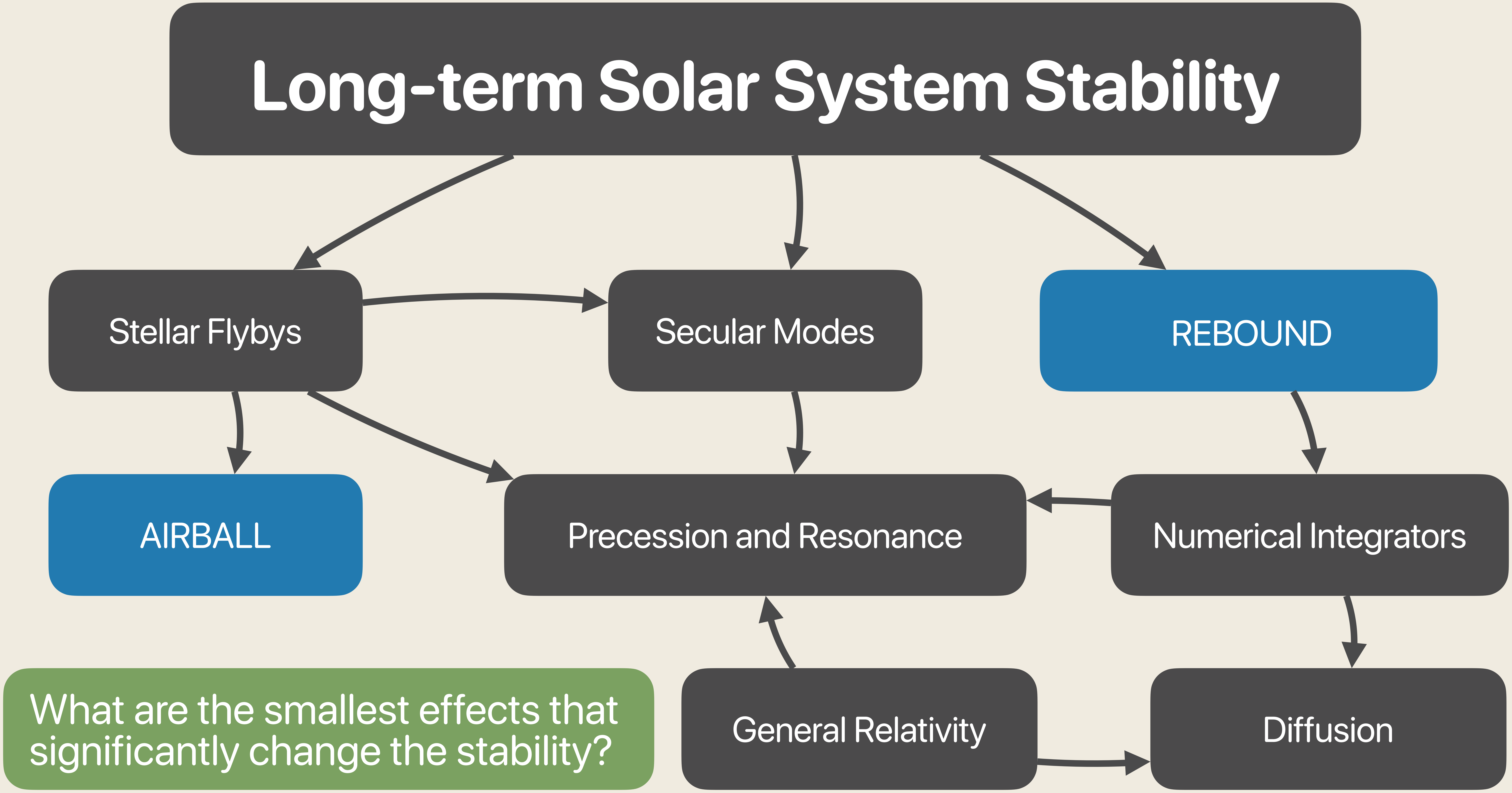
On the Long-term Stability of the Solar System

GARETT BROWN, HANNO REIN

"Those who are interested... must feel some astonishment at seeing how many times the stability of the Solar System has been demonstrated..."

H. Poincaré (1897)




Long-term Solar System Stability
Stellar Flybys
Secular Modes
REBOUND
AIRBALL
Precession and Resonance
Numerical Integrators
What are the smallest effects that significantly change the stability?
General Relativity
Diffusion

The Problem

- Universal Law of Gravitation
- Closed solution for 2-bodies
- No analytical solution for 3+ bodies over infinite time.

1. Perturbative Expansions
2. Numerical Integration

$$\mathbf{F}_{21} = \frac{Gm_1m_2}{|\mathbf{r}_{21}|^2}\hat{\mathbf{r}}_{21}$$

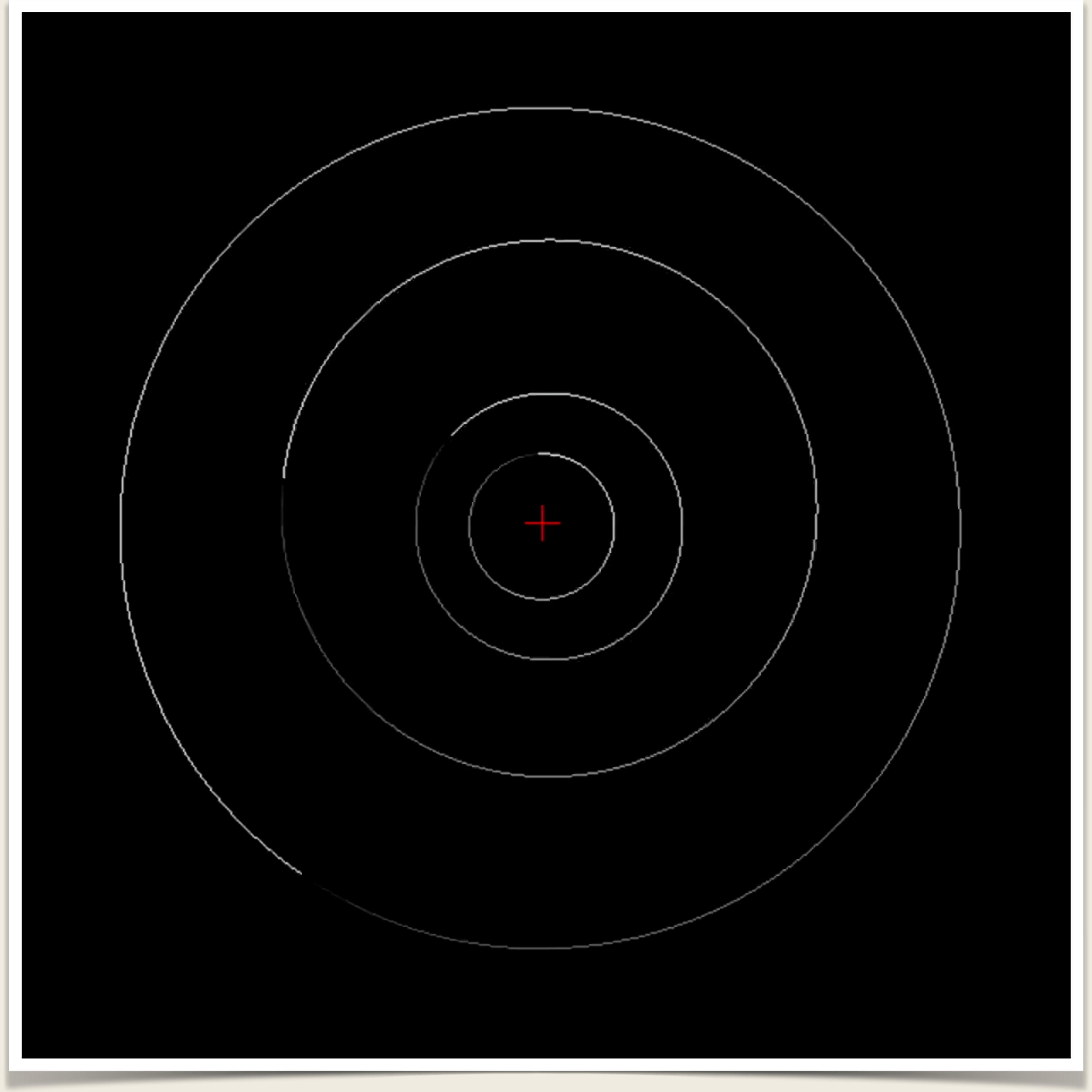
$$\ddot{\mathbf{r}}_i = \sum_{j=1}^{N} m_j \frac{\mathbf{r}_j - \mathbf{r}_i}{\left|\mathbf{r}_j - \mathbf{r}_i\right|^3}, \quad j \neq i$$

$$\mathscr{H} = T + U = \sum_{i=0}^{N-1} \frac{p_i^2}{2m_i} + \sum_{i=0}^{N-1} \sum_{j=i+1}^{N-1} - \frac{Gm_im_j}{|r_i - r_j|}$$

Secular Modes

Average over the fast timescales and expand with perturbation theory...

- Effectively assume the planets are interacting rings of mass
- Mutual interaction causes the orbits to precess and deform
- Linearized equations ⇒ eigenvalues (secular frequencies/modes)
- Quasi-periodic functions

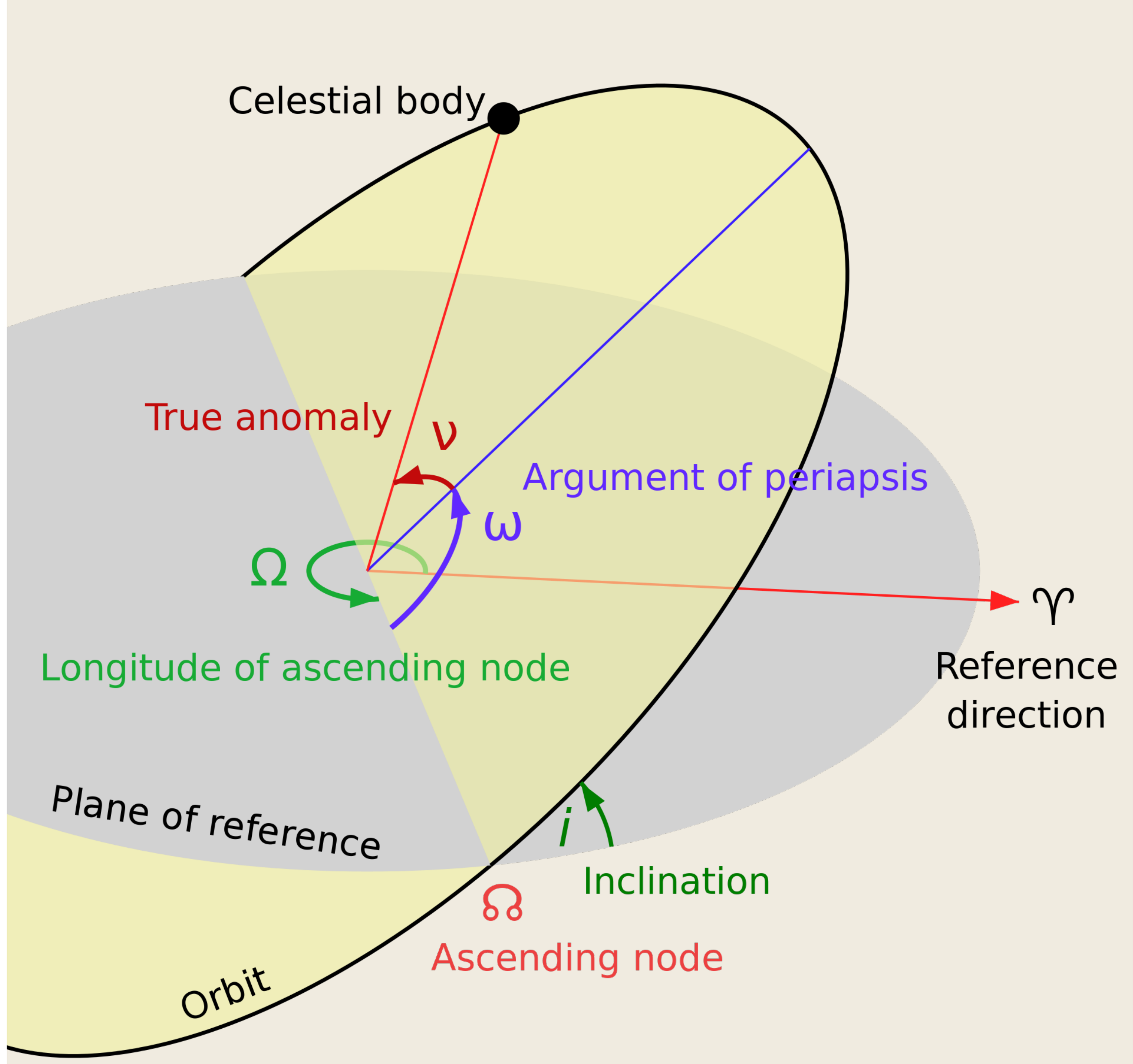


LASKAR (1990) DOI: 10.1016/0019-1035(90)90084-M; ŠIDLICHOVSKÝ & NESVORNÝ (1996) DOI: 10.1007/BF00048443

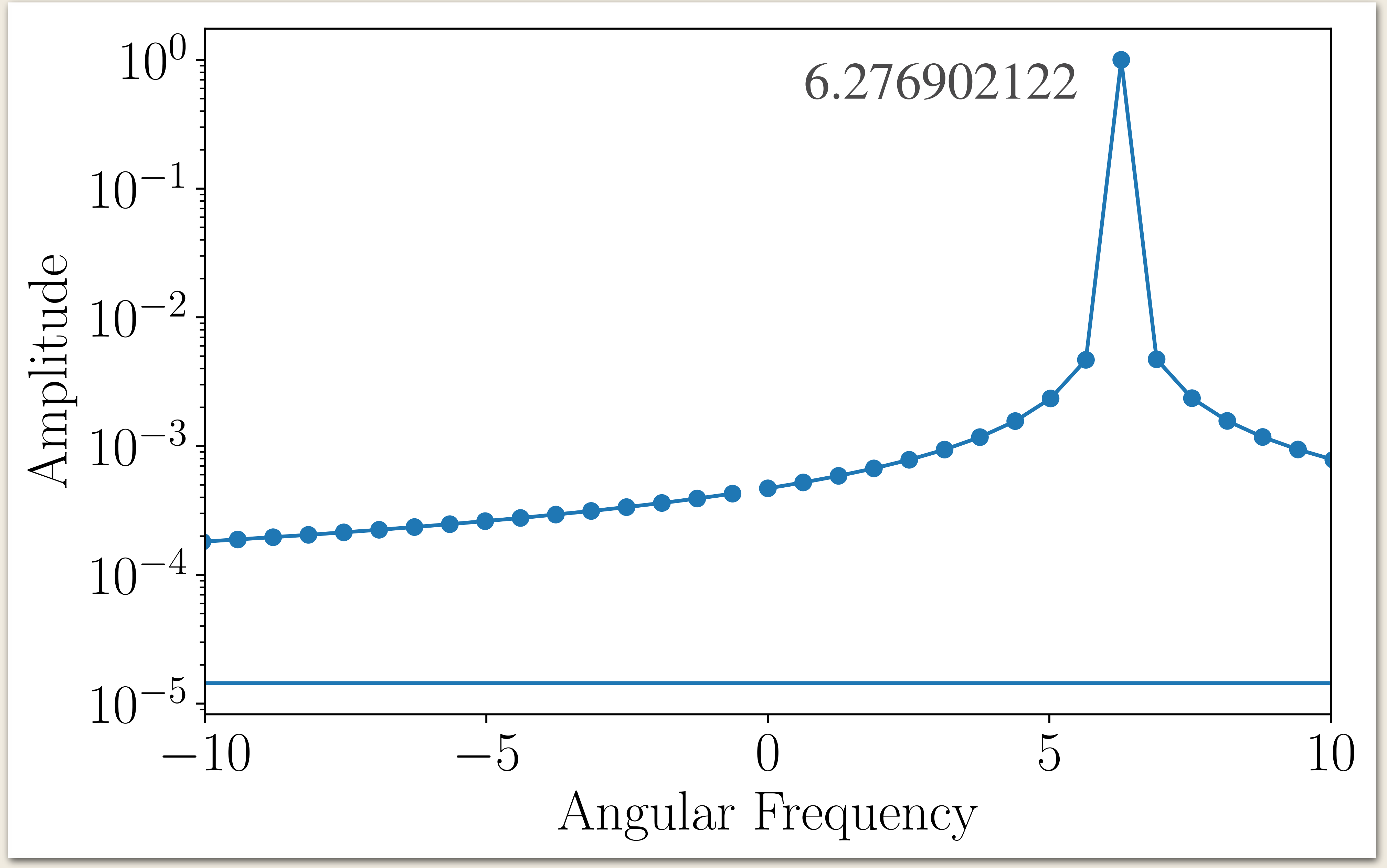
Secular Modes

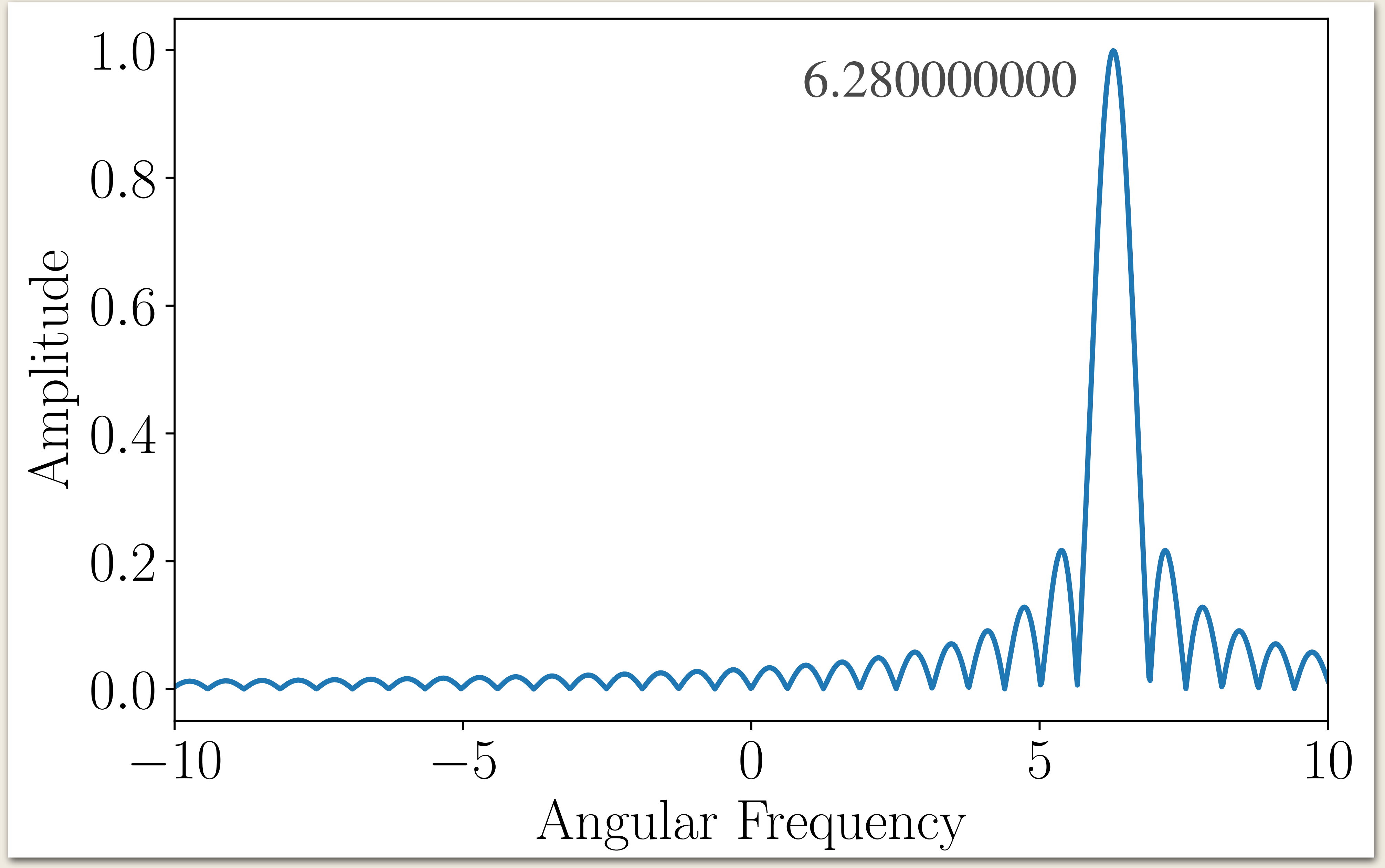
Integrate over millions of years, then find the modes using Modified Fourier Transform (MFT)

- Numerically integrate
- Sample frequently
- Compute the complex eccentricities and inclinations:
  - $z_j(t) = e_j(t)\ \exp[\mathrm{i}\varpi_j(t)]$
  - $\zeta_j(t) = i_j(t)\ \exp[\mathrm{i}\Omega_j(t)]$
- Compute FMFT of $z_j(t)$ and $\zeta_j(t)$.



Wikipedia/Orbital_elements





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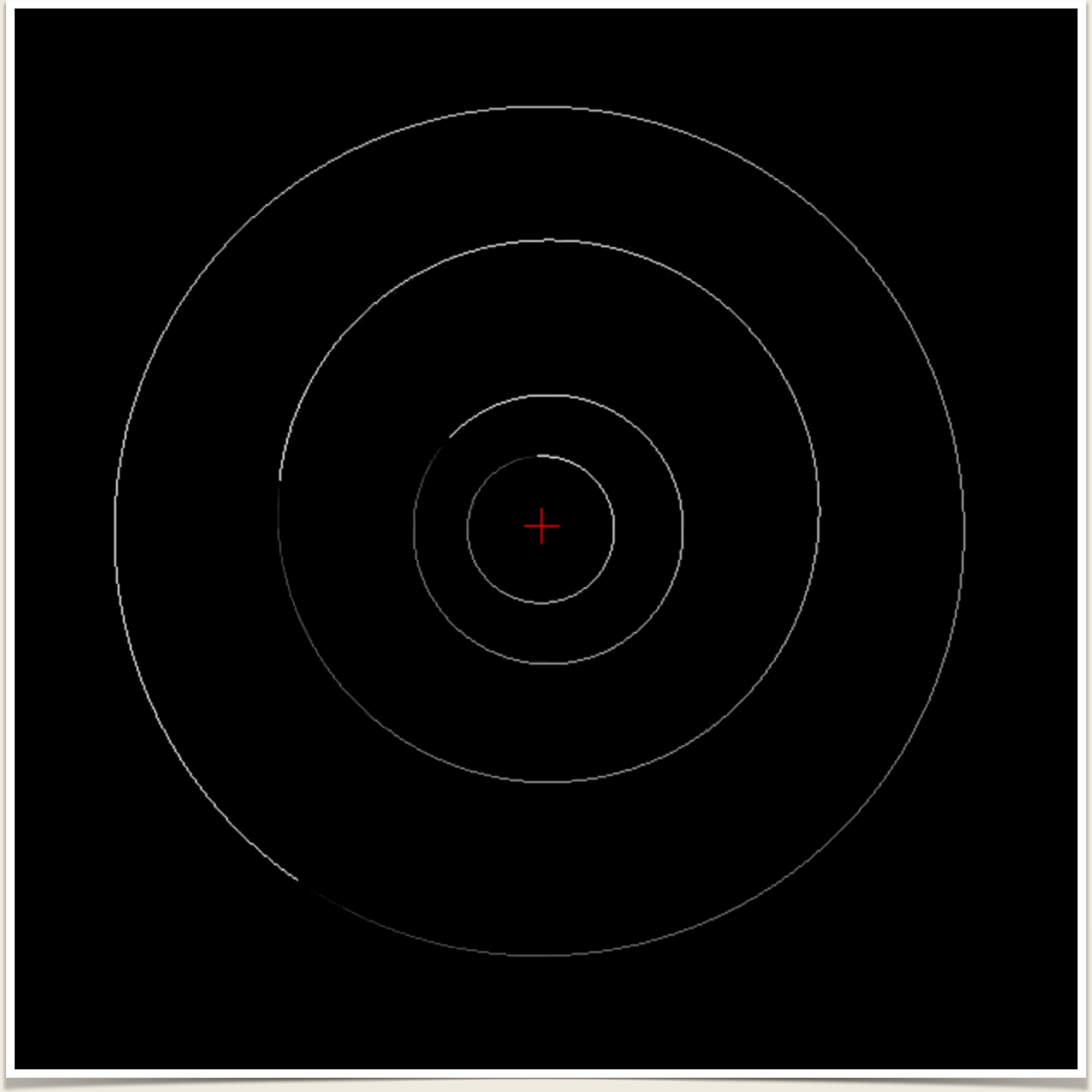
Modified Fourier Transform (MFT)

- Contrived Example
- $y(t) = \exp(6.28\mathrm{i}\, t)$
- $t \in [0,10],\, N = 1000$
- $\phi(\nu) = \langle y(t), \exp(\mathrm{i}\nu t)\rangle$
- Maximize $\phi(\nu)$ in the region of ν.



REIN, LIU (2012) DOI: 10.1051/0004-6361/201118085

REBOUND

- N-body integrator
- Flexible and customizable
- NEW: Real-time, 3D visualization, for both C and Python

<— 700,000 years/second in real time.

REBOUND24: First virtual 2-day REBOUND meeting July 8-9, 2024

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Secular Modes

Integrate over millions of years, then find the modes using Modified Fourier Transform (MFT)

- Solar System fundamental frequencies
- Obtained with modified Fourier transform (MFT)
- 20 Myrs for the inner planets
  50 Myrs for the outer planets

"/yr	Rein & Brown (2019)	Brouwer & van Woerkom's (1950)
g1	5.589	5.463
g2	7.422	7.345
g3	17.3597	17.3283
g4	17.9135	18.0023
g5	4.2575126	4.29591
g6	28.24564	27.77406
g7	3.0879937	2.71931
g8	0.6730304	0.63332

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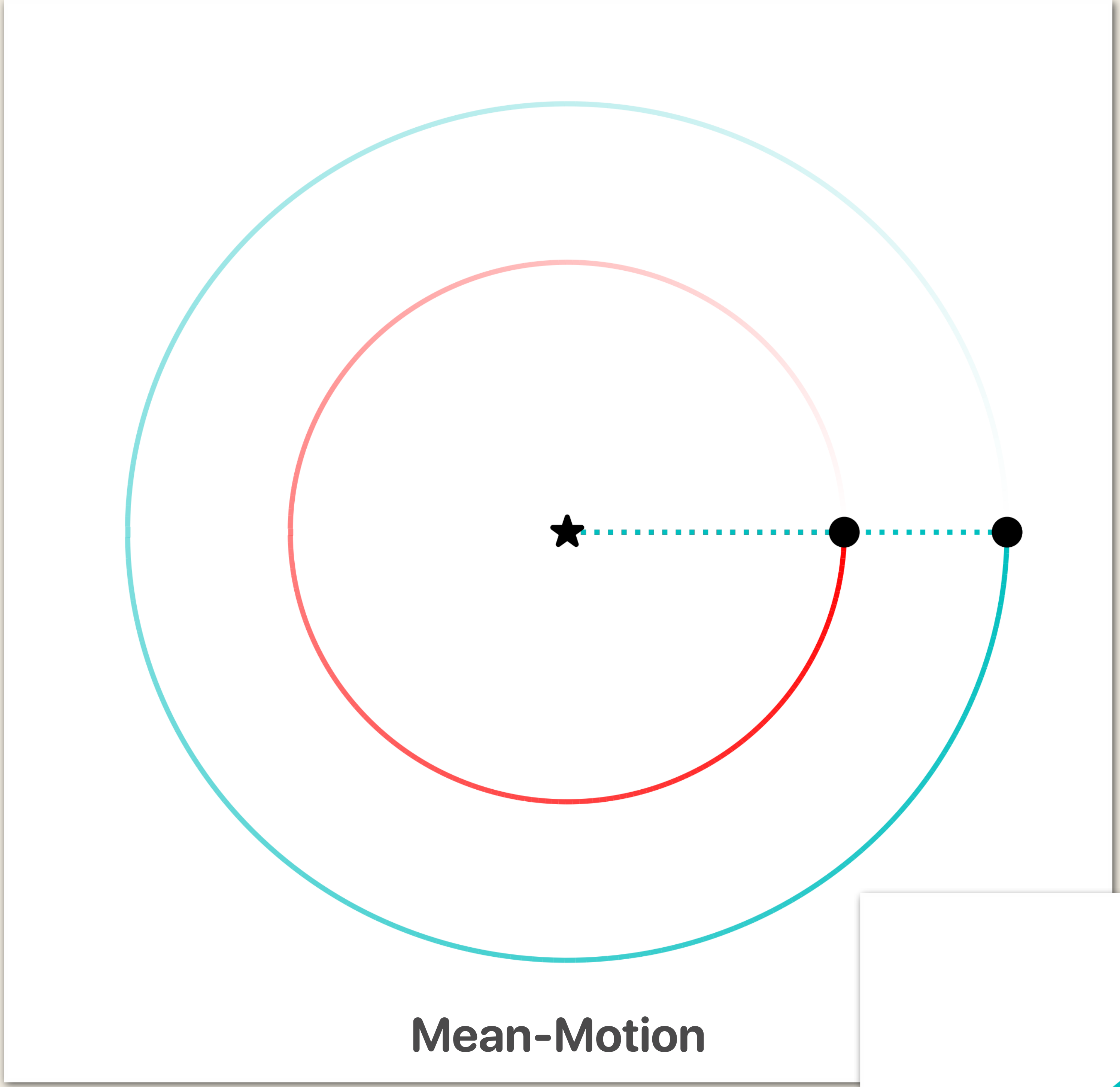
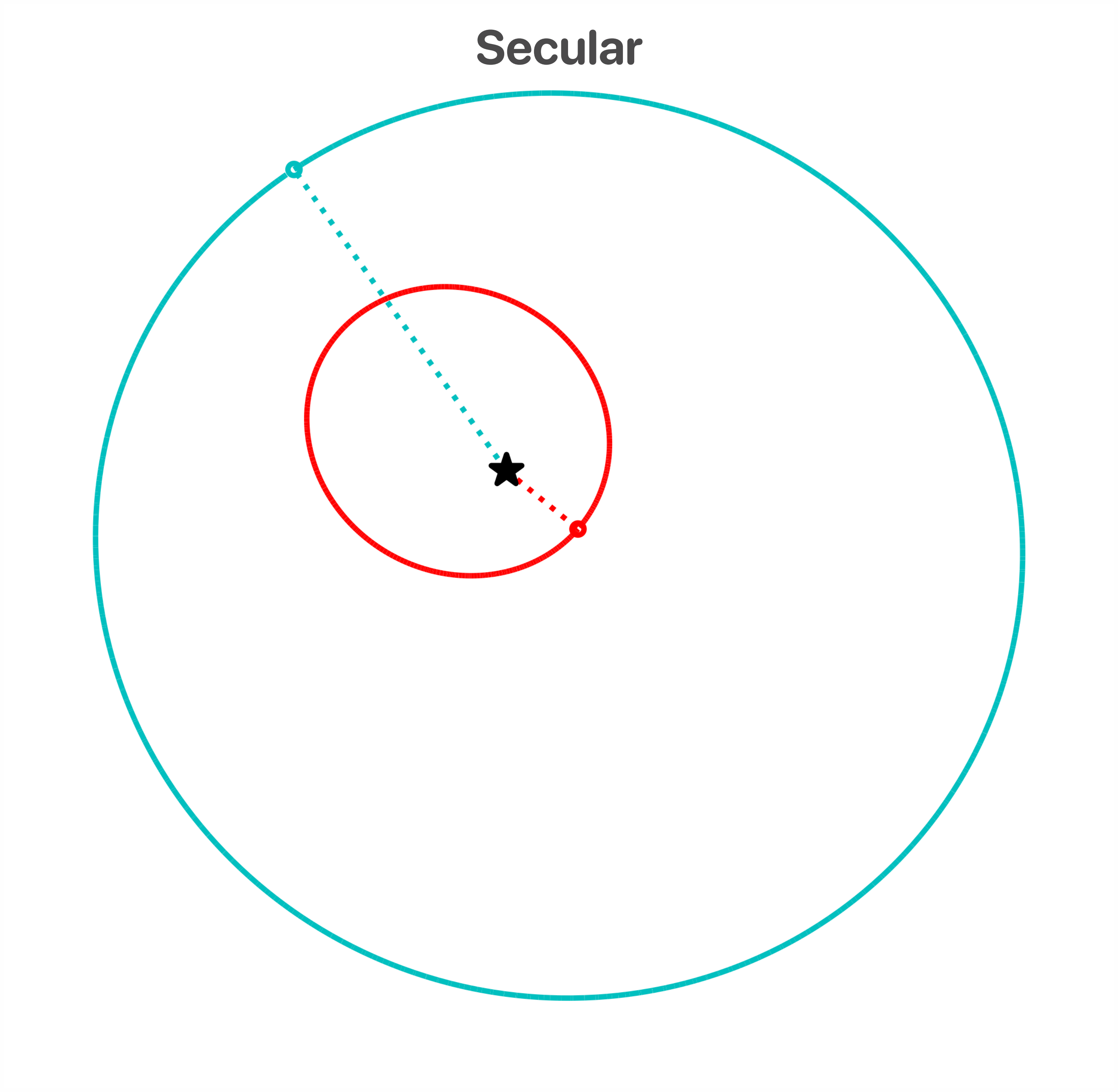




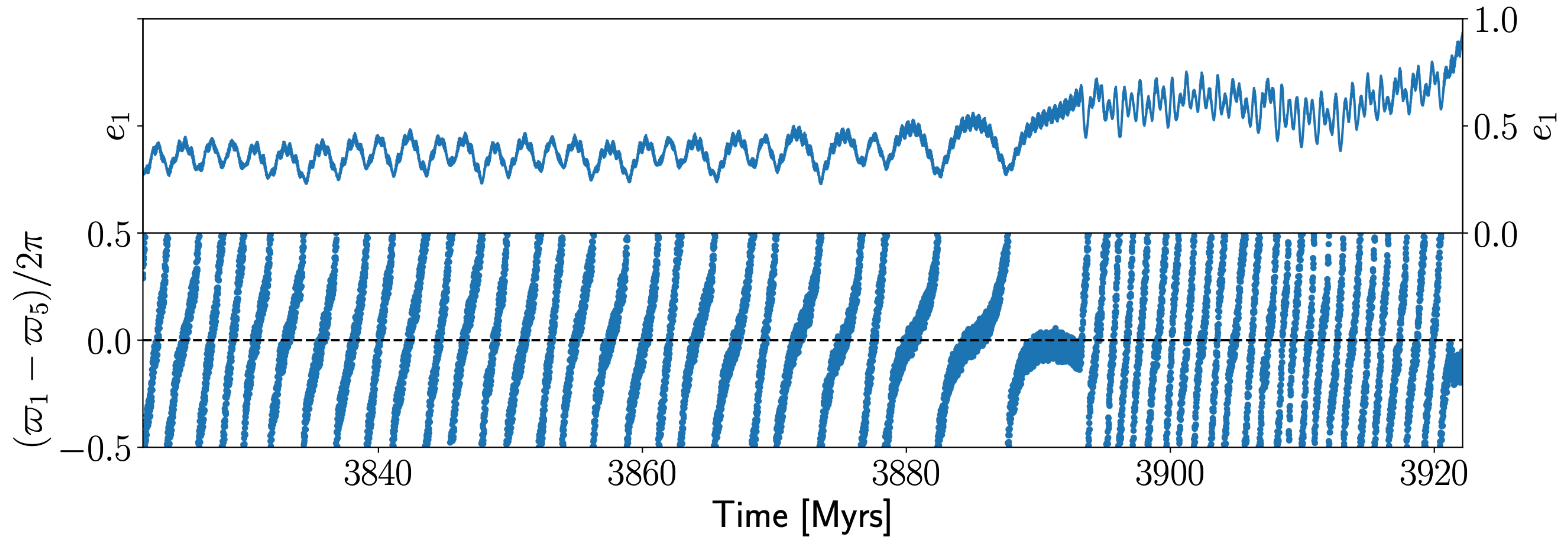
MURRAY & DERMOTT (1999); TREMAINE (2023)

Secular Resonance

- Secular resonances occur when the precessional frequencies of an orbit are similar to another in the system
- Mean-motion resonances occur when the orbital period of one planet is a small integer ratio of another planet's period

LITHWICK & WU (2011) DOI: 10.1088/0004-637X/739/1/31

Secular Resonance



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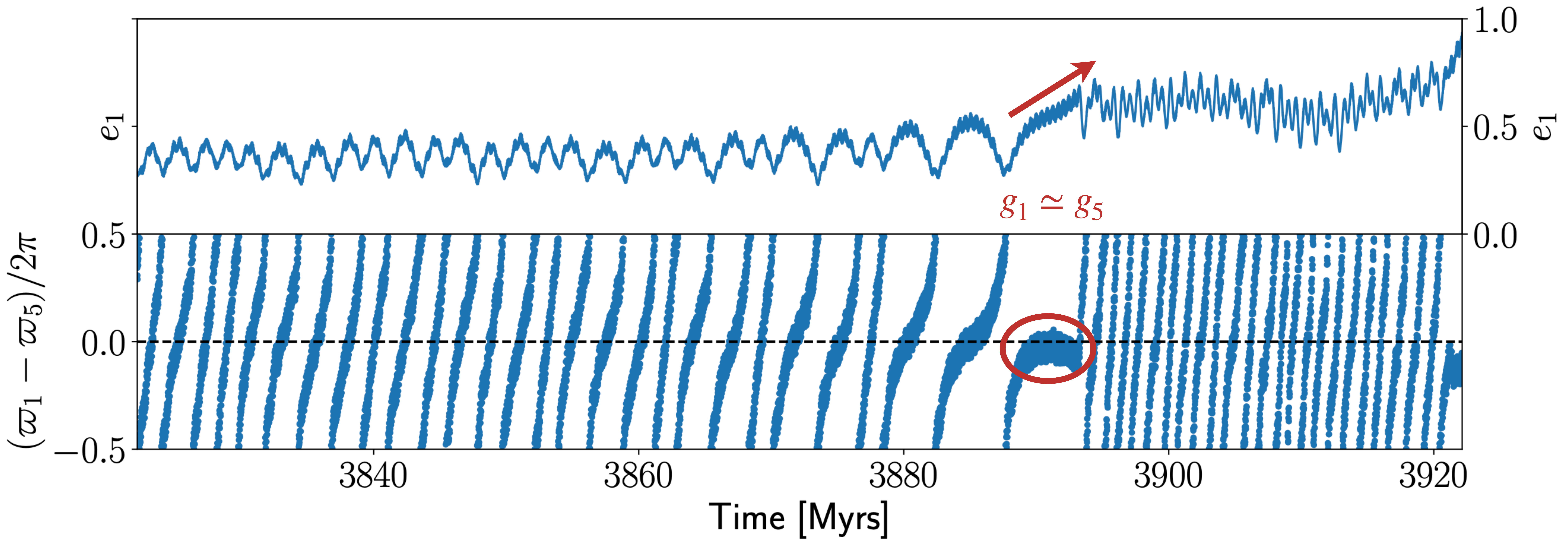
Secular Resonance



A Solution?

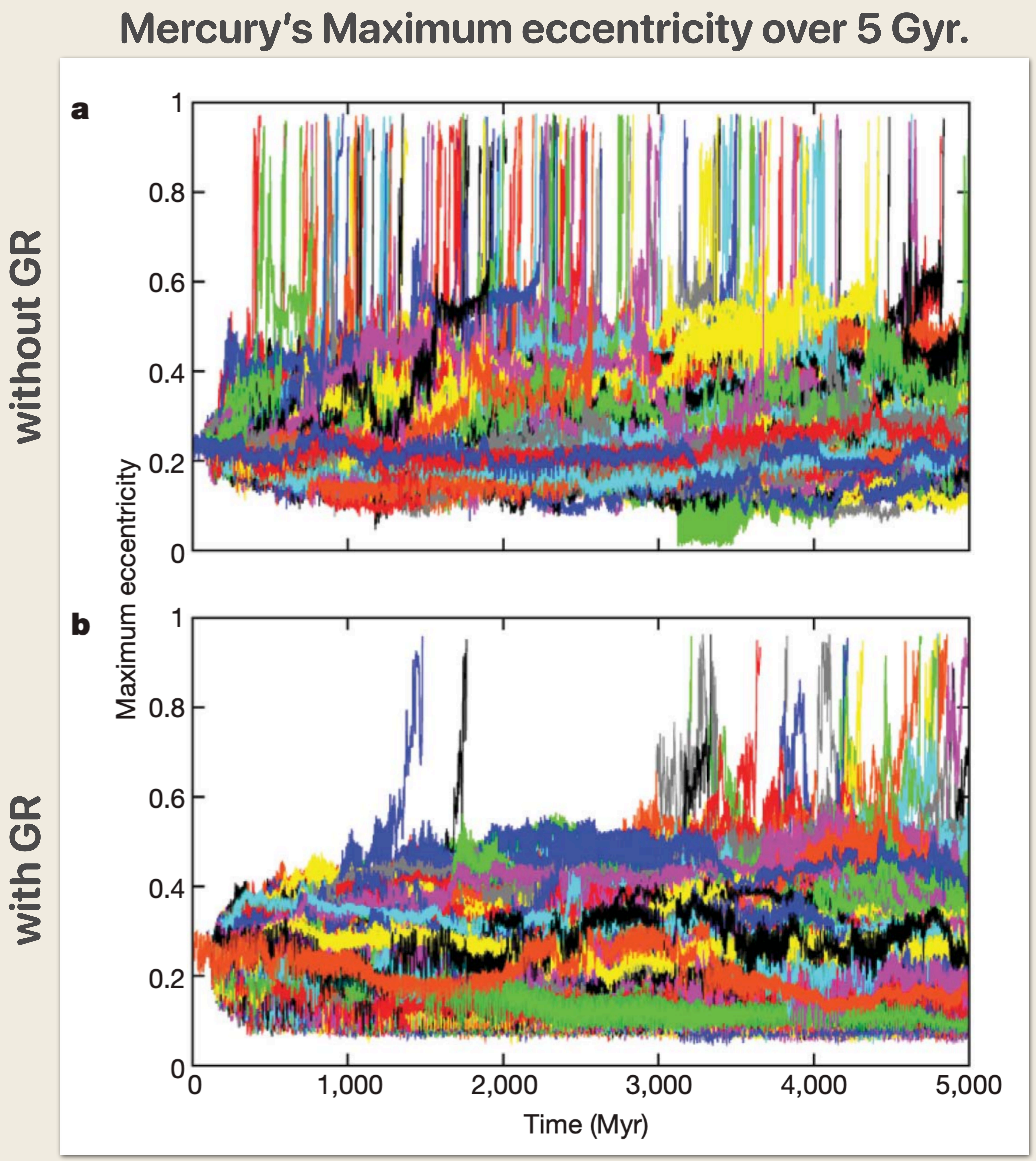
- Split Hamiltonian into "drift" Kepler step ($\mathscr{A}$) and "kick" Interaction step ($\mathscr{B}$)
- Discretize time step, "drift-kick-drift"
- Compute one step at a time
- Time-step needs to be smaller than smallest timescale in the problem
- Need $> 10^{12}$ steps until Sun's red giant phase

$$\mathscr{H} = T + U = \sum_{i=0}^{N-1} \frac{p_i^2}{2m_i} + \sum_{i=0}^{N-1} \sum_{j=i+1}^{N-1} -\frac{Gm_i m_j}{|r_i - r_j|}$$

$$\mathscr{H} = \mathscr{A} + \mathscr{B}$$

$$\mathscr{A} = \sum_{i=0}^{N-1} \frac{p_i^2}{2m_i} + \sum_{i=1}^{N-1} -\frac{Gm_i m_0}{|r_i - r_0|}$$

$$\mathscr{B} = \sum_{i=1}^{N-1} \sum_{j=i+1}^{N-1} -\frac{Gm_i m_j}{|r_i - r_j|}$$



LASKAR & GASTINEAU (2009) DOI:10.1038/NATURE08096

Stability and Chaos

- 201 and 2,501 simulations
- Adjusted the initial semi-major axis of Mercury by at most 0.5 meters
- 60% were unstable without GR
- 1% were unstable with GR (eccentricity of Mercury > 0.9)

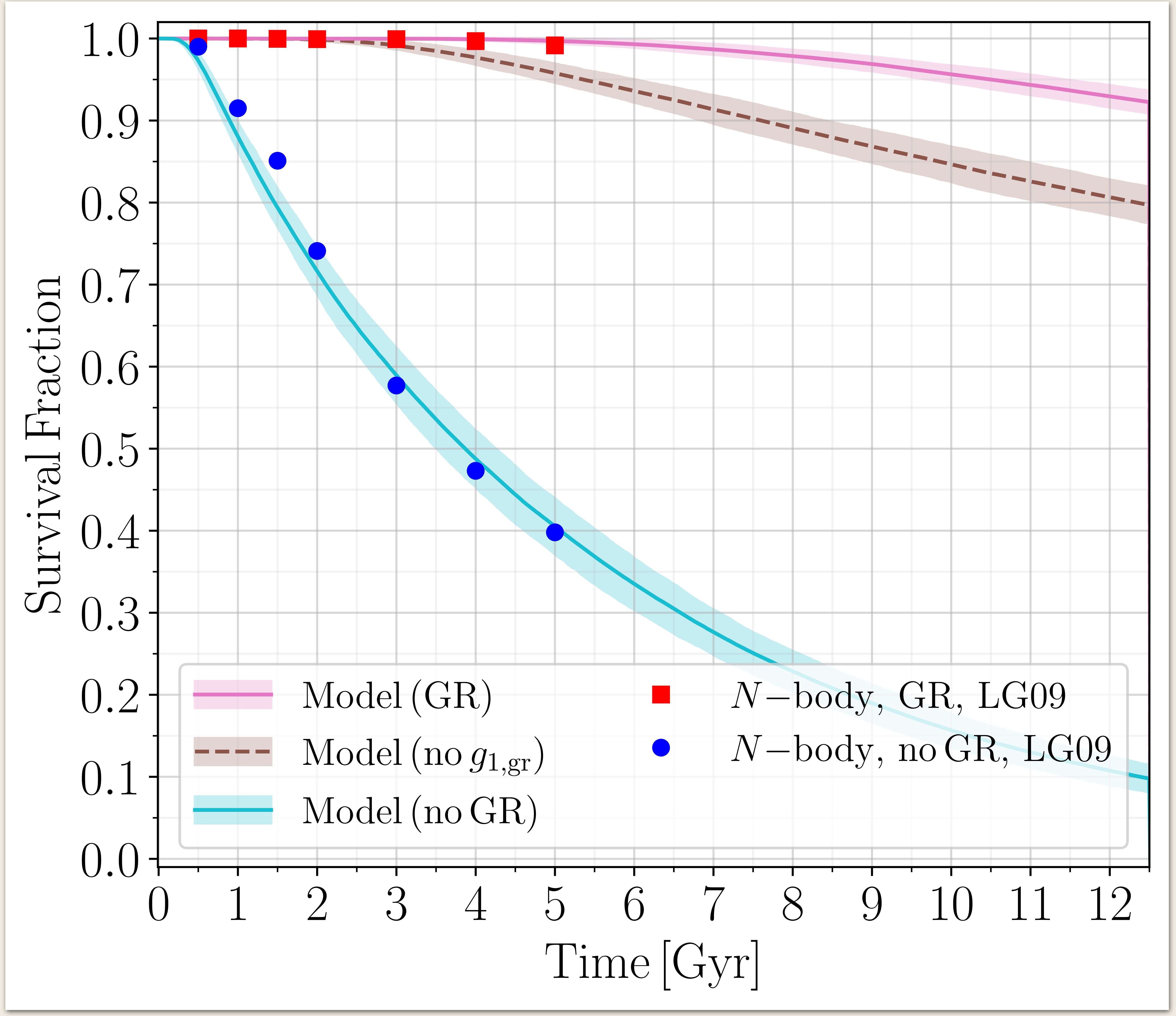
Table 3

Effects	Precession Rate of Perihelion, $\dot{\varpi}$ ($''$/Julian century)
Mercury	0.0050 ± <0.0001
Mercury+Venus Interaction	−0.0053
Venus	277.4176 ± <0.0001
Venus+Earth/Moon Interaction	−0.0209
Venus+Jupiter Interaction	−0.0012
Earth/Moon	90.8881 ± <0.0001
Earth/Moon+Mars Interaction	−0.0016
Mars	2.4814 ± <0.0001
Mars+Jupiter Interaction	0.0002
Jupiter	153.9899 ± <0.0001
Jupiter+Saturn Interaction	0.0411
Saturn	7.3227 ± <0.0001
Saturn+Uranus Interaction	0.0004
Uranus	0.1425 ± <0.0001
Neptune	0.0424 ± <0.0001
Asteroids	0.0012 ± <0.0001
Solar Oblateness	0.0286 ± 0.0011
Gravitoelectric (Schwarzschild-like)	42.9799 ± 0.0009
Lense–Thirring (Gravitomagnetic)	−0.0020 ± 0.0002
Total	575.3100 ± 0.0015

PARK ET AL. (2017) DOI: 10.3847/1538-3881/AA5BE2

Mercury's Perihelion Precession Rate

- Measured as $5.75\,''/\mathrm{yr}$ with $0.43\,''/\mathrm{yr}$ from GR
- But $5.14\,''/\mathrm{yr} \neq 4.26\,''/\mathrm{yr}$

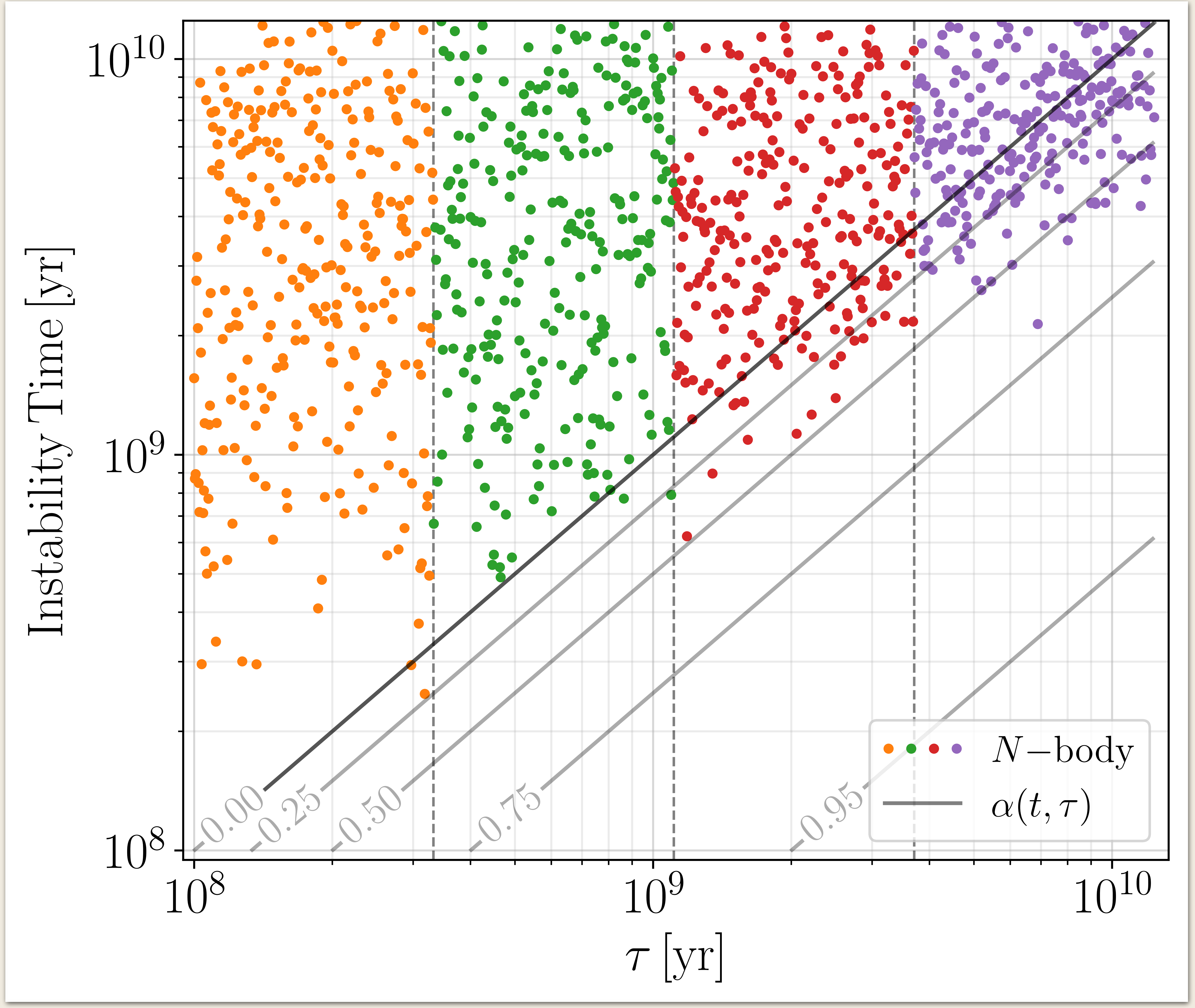


BROWN & REIN (2023) DOI: 10.1093/MNRAS/STAD719
MOGAVERO & LASKAR (2021) DOI: 10.1051/0004-6361/202141007

GR Precession and stability

- Removing only the contribution from GR is insufficient
- The diffusion coefficient must also be altered

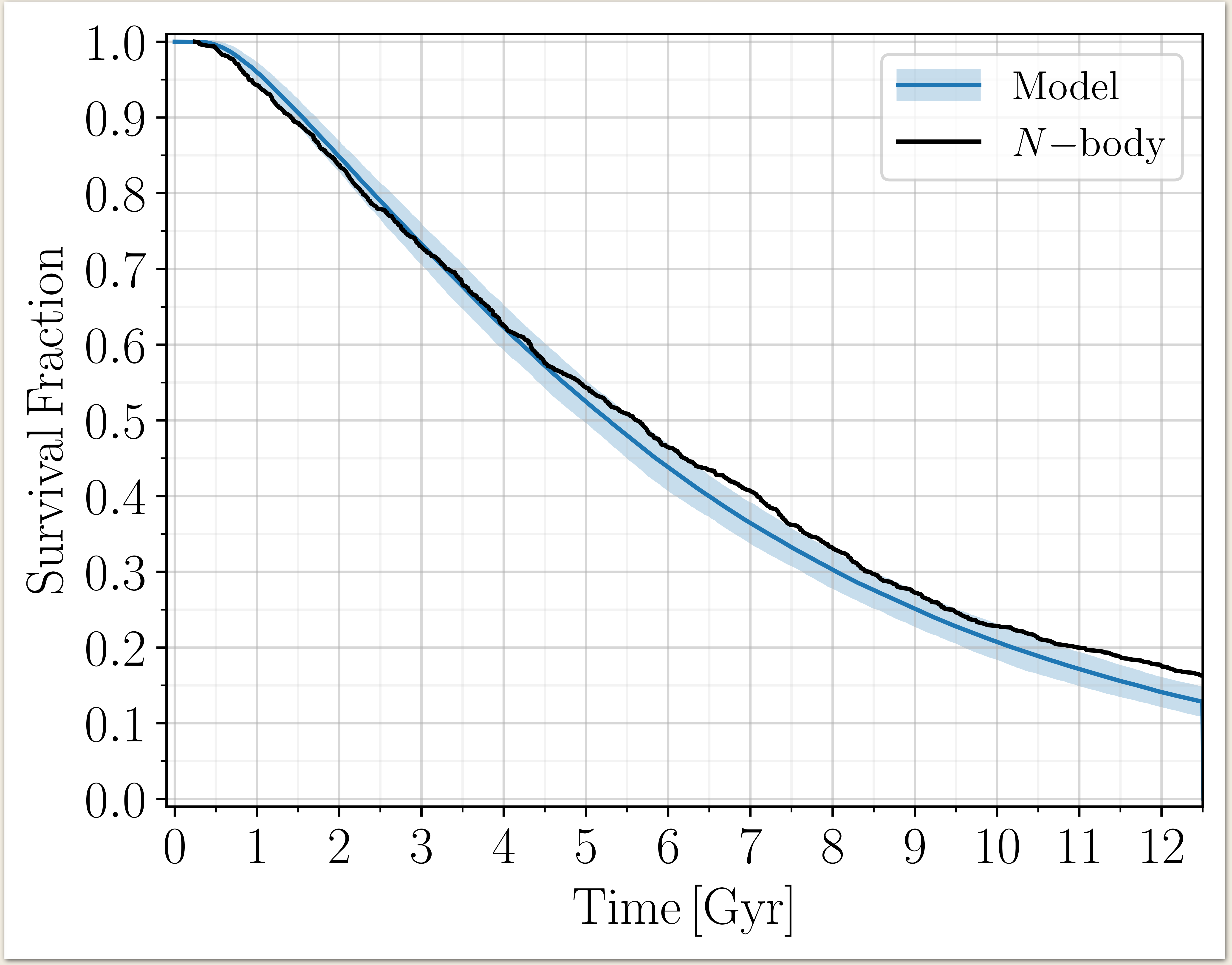
$$\frac{\partial p(g_1, t, \tau)}{\partial t} = -\mu(t, \tau)\frac{\partial p(g_1, t, \tau)}{\partial g_1} + D(t, \tau)\frac{\partial^2 p(g_1, t, \tau)}{\partial g_1^2}$$



BROWN & REIN (2023) DOI: 10.1093/MNRAS/STAD719

GR Precession and stability

- 1% instability with GR
- 60% instability without GR
- What if GR "turns off" over millions of years?
- Smooth interpolation



BROWN & REIN (2023) DOI: 10.1093/MNRAS/STAD719
ABBOT ET AL. (2023) DOI: 10.48550/ARXIV.2306.11870

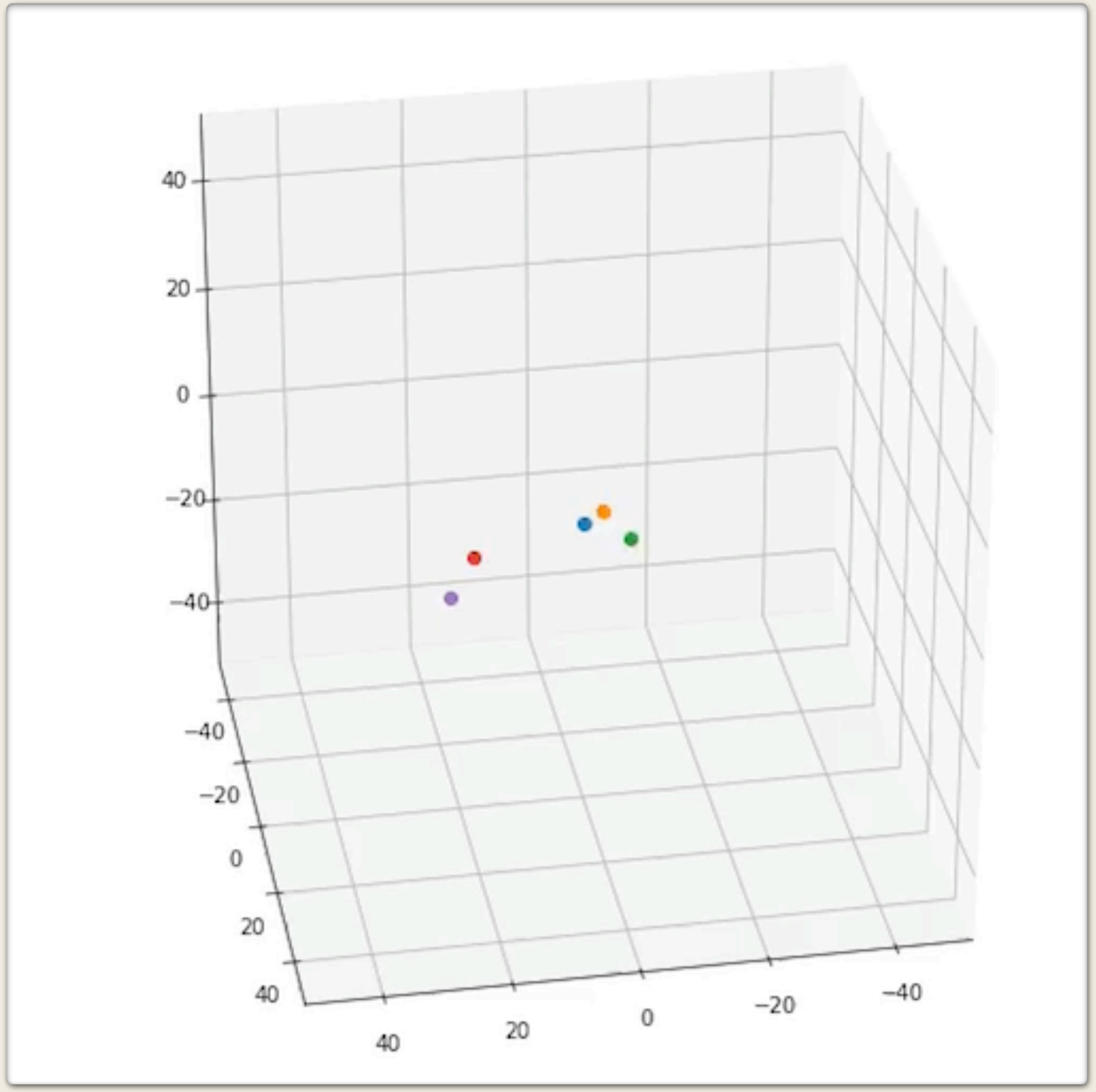
GR Precession and stability

- So long as physical diffusion dominates the dynamics, numerical integration can be trusted
- Further improvement for $P(t < 5\,\mathrm{Gyr})$ with a subdiffusive model, Abbot et al. (2023)

HEGGIE (1975) DOI: 10.1093/MNRAS/173.3.729

Stellar Flybys

- A passing star, external to the system of interest
- Often very distant and unnoticeable
- Can be very violent and destructive
- Is there a middle ground?

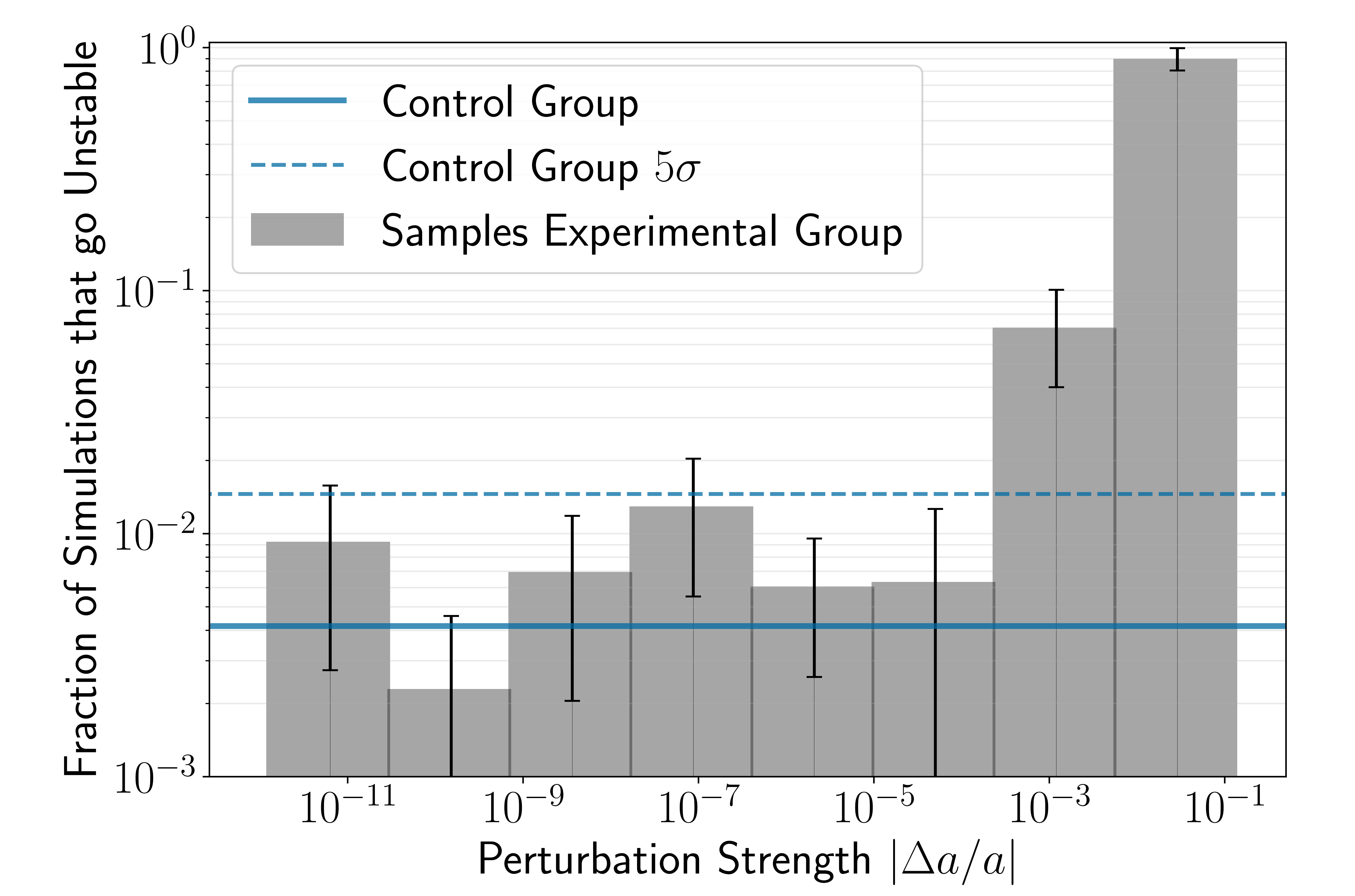


BROWN & REIN (2022) DOI: 10.1093/MNRAS/STAC1763

Experimental Setup: Stellar Flybys

- 2,880, 4.8 Gyrs solar system simulations:
  - 960 control sims, 1920 experimental sims
- NASA JPL Horizons, J2000 epoch
- REBOUND N-body integrator
- REBOUNDx with `gr_potential`
- WHCKL integrator
- `dt = 8.062` days

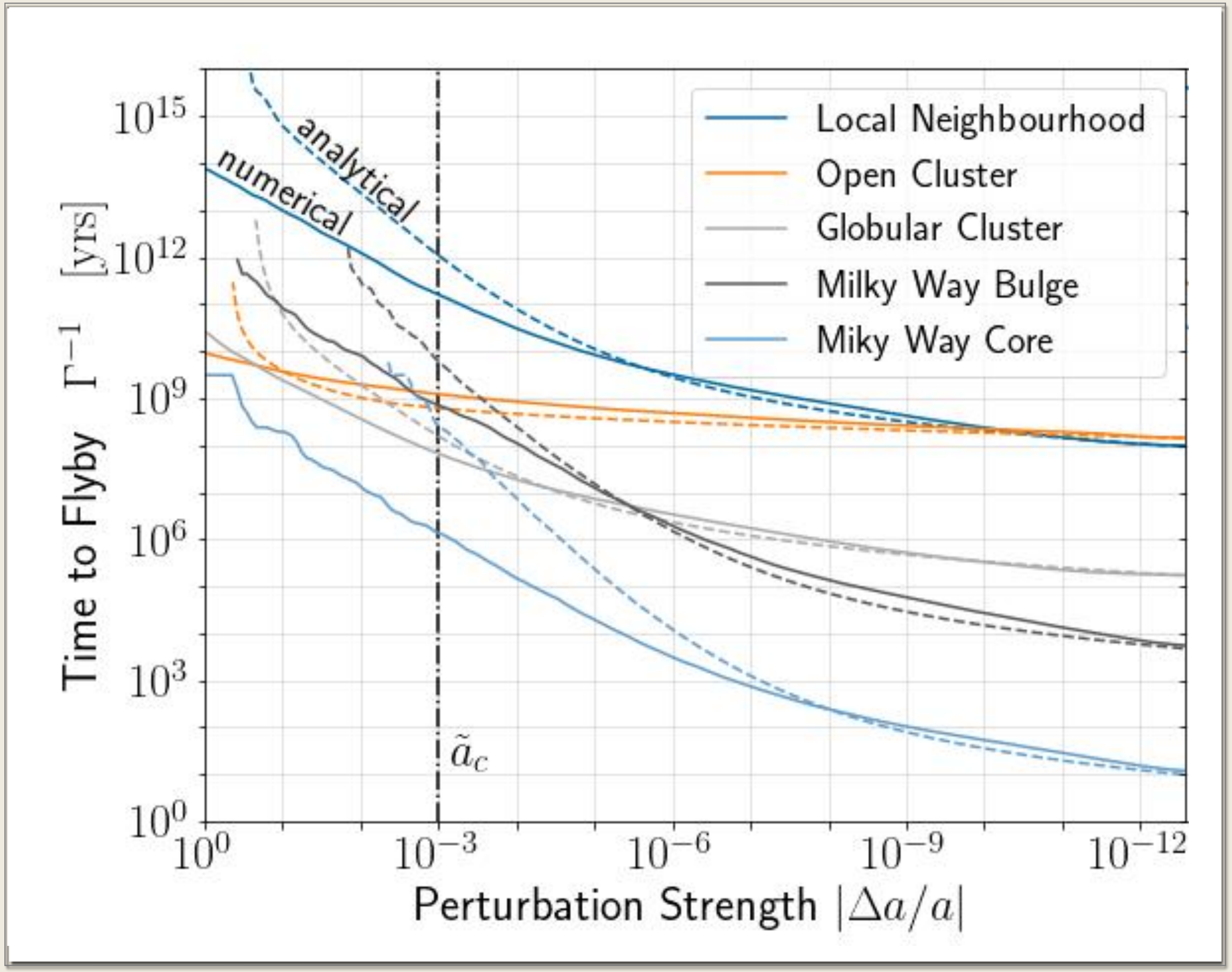
Stellar flybys affect Neptune's semi-major axis more than the other planets



BROWN & REIN (2022) DOI: 10.1093/MNRAS/STAC1763

Stellar Flybys and stability

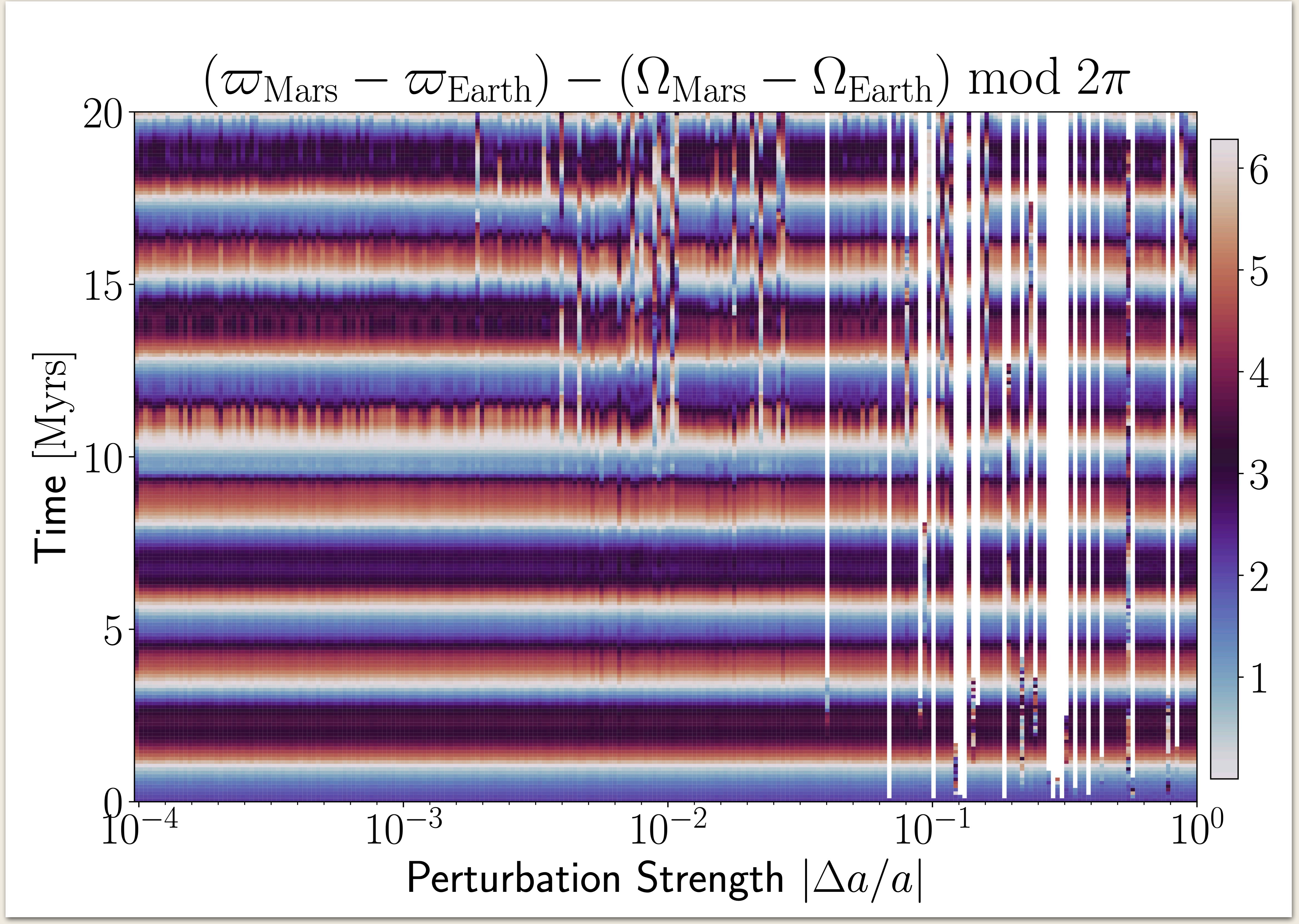
- 0.1% relative change to Neptune's semi-major axis increases instability within 5 Gyr by one order of magnitude.



BROWN & REIN (2022) DOI: 10.1093/MNRAS/STAC1763

Stellar Flybys and stability

- A flyby of this magnitude is not likely to pass the solar system for 100 Gyr



BROWN & REIN (2022) DOI: 10.1093/MNRAS/STAC1763

Stellar Flybys and stability

- Changes to Neptune's orbit effect the inner solar system secular state
- Angular momentum is exchanged between the planets on secular timescales

BROWN ET AL. (IN PREP) AIRBALL.READTHEDOCS.IO

AIRBALL

- A package for running and managing flybys in REBOUND
- Stellar environments for generating flyby stars
- Initial mass functions
- `astropy.units`

BROWN ET AL. (IN PREP)

AIRBALL.READTHEDOCS.IO

AIRBALL

```
> import rebound
> import airball
> sim = rebound.Simulation()
> sim.add('solar system')
> stars = airball.Stars(airball.OpenCluster, size=10)
> airball.flybys(sim, stars)
```

Summary

- Modern computation provides insights into Newton's centuries old problem.
- General Relativity is essential to solar system stability over its lifetime.
- The secular diffusion of Mercury through phase space leads to instability when Mercury enters a linear secular resonance with Jupiter.
- Even weak stellar flybys can increase the chance of solar system instability.
-